Geometric Measure Theory and its Applications

6/4/2007

In this final lecture of the UCLA/LANL (short) course on GMT and applications, I will look at measure theoretic densities.

A (Possibly Peculiar) Motivation for looking at densities:

If $f \in L^1_{loc}(U)$ we say that $g_i \in L^1_{loc}(U)$ is the weak partial derivative w.r.t. x_i if

$$\int_U f \frac{\partial \phi}{\partial x_i} dx = -\int_U g_i \phi \, dx \qquad \forall \phi \in C_c^1(U)$$

A remark like: "it is easy to see that g_i is unique if it exists" is usually made now. Why? Because

$$\int_U (g_i - \tilde{g}_i)\phi \, dx = 0 \quad \forall \phi \in C_c^1(U) \Rightarrow g_i - \tilde{g}_i = 0 \quad \mathcal{L}^n \text{ a.e.}$$

But why is this true? Because of an easy approximation and the Lebesgue-Besicovitch differentiation theorem... and that is where the densities come into the picture.

Lebesgue-Besicovitch differentiation theorem

Let μ be a Radon measure on $\mathbb{R}^n$ and $f \in L^1_{loc}(\mathbb{R}^n, \mu)$.

Then

$$\lim_{r \to 0} \fint_{B(x,r)} f \, d\mu = f(x) \qquad \mu \text{ a.e. } x \in \mathbb{R}^n$$

where

$$\fint_E f \, d\mu \equiv \frac{1}{\mu(E)} \int_E f \, d\mu$$

In our case we let $\mu = \mathcal{L}^n$. This gives us

$$\frac{1}{\alpha(n) r^n} \int_{B(x,r)} f \, d\mathcal{L}^n = f(x) \quad \mathcal{L}^n \text{ a.e. } x$$

Now we approximate: For any ball $B(x,r) \in \mathbb{R}^n$ we approximate $\chi_{B(x,r)}$, the characteristic function of the set $B(x,r)$ by a smooth function $h_\varepsilon \ni h_\varepsilon(y) = 0$ for $|y-x| \geq r+\varepsilon$, $h_\varepsilon(y) = 1$ for $|y-x| \leq r$, and $0 \leq h_\varepsilon \leq 1$ on $\mathbb{R}^n$. Clearly all the h_ε's are in $C_c^1(U)$ (for the balls we are concerned with).

choose an $x \in \mathbb{R}^n$, $r \in \mathbb{R}^+$. $\displaystyle\int h_\varepsilon (g_i - \tilde{g}_i)\, dx = 0 \quad \forall \varepsilon$

but $\displaystyle\int h_\varepsilon (g_i - \tilde{g}_i)\, dx \longrightarrow \int_{B(x,r)} (g_i - \tilde{g}_i)\, dx \Rightarrow \int_{B(x,r)} (g_i - \tilde{g}_i)\, dx = 0$

$$\Rightarrow \lim_{r \to 0} \frac{1}{\alpha(n) r^n} \int_{B(x,r)} (g_i - \tilde{g}_i) = 0$$

using Lebesgue - Besicovitch $\Rightarrow g_i - \tilde{g}_i = 0 \quad \mathcal{L}^n$ a.e. in $\mathbb{R}^n$.

Let's look at the Lebesgue - Besicovitch differentiation theorem a little more closely.

again:

Let μ be a Radon measure on $\mathbb{R}^n$ and $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, d\mu = f(x) \quad \mu \text{ a.e. } x \in \mathbb{R}^n$$

Remark: The proof uses simple facts about the differentiation of Radon measures.

We can use the Lebesgue - Besicovitch differentiation theorem to say something about subsets of sets with finite Hausdorff measure. Let $H^k(E) < \infty$. Define $\mu \equiv H^k \llcorner E$. For any $F \subset E$ we get

$$\lim_{r \to 0} \frac{H^k(F \cap B(x,r))}{H^k(E \cap B(x,r))} = \lim_{r \to 0} \fint_{B(x,r)} \chi_F \, d\mu = 1 \quad \mu \text{ a.e. } x \in F$$

$$= 0 \quad \mu \text{ a.e. } x \in \mathbb{R}^n - F$$

We cannot of course say anything about densities of points in E w.r.t the measure $\mu = H^k \llcorner E$. I.E letting $\mu = H^k \llcorner E$ and $f = \chi_E$ we get the trivial statement that

$$\lim_{r \to 0} \frac{H^k(E \cap B(x,r))}{H^k(E \cap B(x,r))} = 1 \quad H^k \llcorner E \text{ a.e. } x \in E$$

Actually there is a little bit of information here. This tells us that the H^k measure of points $\hat{x}$ in $E \ni H^k(E \cap B(x,r)) = 0$ for some $r > 0$ is 0. But much more can be gained by considering the following densities.

$$\lim_{r \to 0} \frac{H^k(E \cap B(x,r))}{\alpha(k) r^k}$$

Of course we do not know that this limit exists so we will usually start with

$$\liminf_{r \to 0} \frac{H^k(E \cap B(x,r))}{\alpha(k) r^k} \quad \text{and} \quad \limsup_{r \to 0} \frac{H^k(E \cap B(x,r))}{\alpha(k) r^k}$$

Theorem: if $E \subset \mathbb{R}^n$, E is H^k measurable, $H^k(E) < \infty$ then

$$\lim_{r \to 0} \frac{H^k(E \cap B(x,r))}{\alpha(k) r^k} = 0$$

for H^k a.e. $x \in \mathbb{R}^n - E$

Proof: (from Evans and Gariepy)

Define

$$A_t \equiv \left\{ x \in \mathbb{R}^n - E \;\middle|\; \limsup_{r \to 0} \frac{H^k(B(x,r) \cap E)}{\alpha(k) r^k} > t \right\}$$

Since $H^k \llcorner E$ is Radon we can find compact $K_\varepsilon \subset E \ni$

$$H^k(E - K_\varepsilon) < \varepsilon$$

as long as $\varepsilon > 0$. Define $U_\varepsilon \equiv \mathbb{R}^n - K_\varepsilon$. U_ε is open. $A_t \subset U_\varepsilon$.

Choose $\delta > 0$. Define

$$\mathcal{F} \equiv \left\{ B(x,r) \;\middle|\; B(x,r) \subset U_\varepsilon,\ 0 < r < \delta,\ \frac{H^k(B(x,r) \cap E)}{\alpha(k) r^k} > t \right\}.$$

By the Vitali Covering Theorem, there exists a countable disjoint family of balls $\{B(x_i, r_i)\}_{i=1}^{\infty}$ in $\mathcal{F} \ni A_t \subset \bigcup_{i=1}^{\infty} B(x_i, 5r_i)$. Then

$$H^k_{10\delta}(A_t) \le \sum_{i=1}^{\infty} \alpha(k)(5r_i)^k < \frac{5^k}{t} \sum_{i=1}^{\infty} H^k(B(x_i, r_i) \cap E)$$

$$\le \frac{5^k}{t} H^k(U_\varepsilon \cap E)$$

$$\le \frac{5^k \varepsilon}{t}$$

Since δ is arbitrary, we obtain that $H^k(A_t) \le \frac{5^k \varepsilon}{t}$, but ε was arbitrary so $H^k(A_t) = 0 \quad \forall\, t > 0$. This implies

$$\limsup_{r \to 0} \frac{H^k(B(x,r) \cap E)}{\alpha(k) r^k} = 0 \quad H^k \text{ a.e. } x \in \mathbb{R}^n - E \Rightarrow \text{the theorem.}$$

What about the density of points in E? If nothing is known about E except that it has finite H^k measure then we have the following result.

Theorem

If $E \subset \mathbb{R}^n$, E is H^k measurable, $H^k(E) < \infty$ then

$$\frac{1}{2^k} \le \limsup_{r \to 0} \frac{H^k(B(x,r) \cap E)}{\alpha(k)\, r^k} \le 1$$

H^k a.e. x in E.

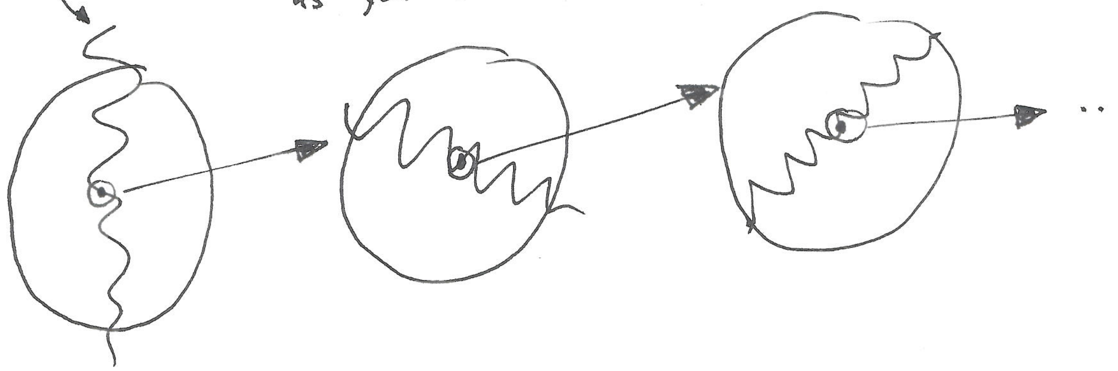
Remarks: ① Notice this says nothing about how often

$$\lim_{r \to 0} \frac{H^k(B(x,r) \cap E)}{\alpha(k)\, r^k} \text{ exists.}$$

② This tells us that fractal like curves must have infinite length.

Suppose a curve γ keeps havg wiggles as you zoom in:

γ



Then $$\limsup_{r \to 0} \frac{H^k(B(x,r) \cap E)}{\alpha(k)\, r^k} > 1$$ (in this picture of course $k = 1$)

$\Rightarrow$ $H^k(E)$ is infinite.

Rectifiability Regularity Theorem

We next state and prove a regularity theorem for rectifiable sets. To do this we need the notion of approximate (measure theoretic) tangent spaces.

Definition: let $\eta_{x,\lambda}(y) = \frac{y-x}{\lambda}$. Then $(\eta_{x,\lambda})_{\#}(E)$ is the x-centered $\frac{1}{\lambda}$-dilation of E, for $E \subset \mathbb{R}^n$.

Definition: if E is an H^k measurable subset of $\mathbb{R}^n$ then P - any k dim subspace of $\mathbb{R}^n$ - is the approximate tangent space of E at x if

$$\lim_{\lambda \downarrow 0} \int_{(\eta_{x,\lambda})_{\#}(E)} f(y)\, dH^k = \int_P f(y)\, dH^k \quad \forall f \in C_c^0(\mathbb{R}^n)$$

Remark: The above definition actually requires that $H^k(E \cap K) < \infty$ for compact $K \subset \mathbb{R}^n$. We generalize now by allowing a density $\theta(x)$ to be used to get the "integrability" of E. i.e.

$\int_{E \cap K} \theta(x)\, dH^k < \infty$ for K compact in $\mathbb{R}^n$.

Definition: If E is an H^k measurable subset of $\mathbb{R}^n$ and θ is a positive locally H^k-integrable function on E, then we say that P is an approximate tangent space for E at x relative to θ if

$$\lim_{\lambda \downarrow 0} \int_{(\eta_{x,y})_{\#}(E)} f(y)\, \theta(x+\lambda y)\, dH^k = \theta(x) \int_P f(y)\, dH^k$$

for all $f \in C_c^0(\mathbb{R}^n)$.

Now for the regularity result.

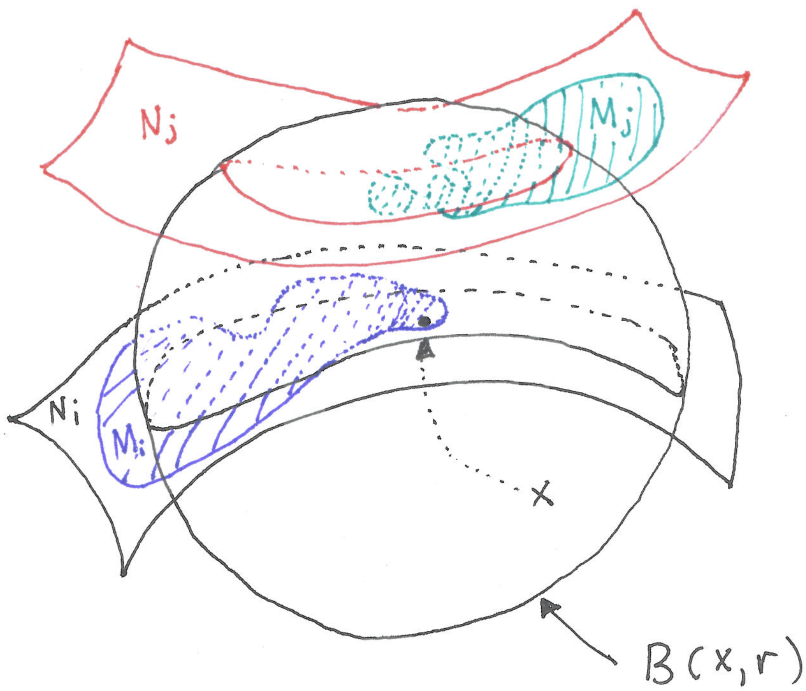
<u>Theorem:</u> Suppose E is H^k measurable. Then

E is countably k-rectifiable

$\Updownarrow$

$\exists$ a positive locally H^k-integrable function θ with respect to which an approximate tangent space $P(x)$ exists for H^k a.e. $x \in E$.

<u>Proof:</u> $\Downarrow$)



First, use the C^1 submanifold representation of a rectifiable set to get

$$E = \bigcup_{i=0}^{\infty} M_i \quad , \quad M_i \cap M_j = \emptyset \ i \neq j, \ H^k(M_0) = 0$$

$M_i \subset N_i$ $i = 1, 2, \ldots$ N_i a C^1 k-dim embedded submanifold of $\mathbb{R}^n$

Without loss of generality, we assume that $H^k(M_i) \leq 1 \ \forall i$

Next define $\theta(y) = \theta_i \equiv \frac{1}{2^i}$ for $y \in M_i$ and 0 elsewhere.
Now define $\mu = H^k \llcorner \theta$ ($= \theta H^k \llcorner E$). Notice that μ is Borel Regular and finite, therefore also Radon.

Now pick $x \in M_i$ $\ni$

$$\frac{H^k(B(x,r) \cap M_i)}{H^k(B(x,r) \cap N_i)} \xrightarrow[r \to 0]{} 1$$

we know that the set of $x \ni$ this holds is μ almost all of M_i and therefore H^k almost all of M_i.

Pick any $f \in C_c^0(\mathbb{R}^n)$ and let $B(0, R)$ contain the support of f. Suppose $|f| \le C_f$.

$$\underbrace{\int_{\eta_{x,\lambda\#}(E)} f(y)\,\theta(x+\lambda y)\,dH^k}_{*} = \frac{1}{\lambda^k}\int_E f\left(\frac{z-x}{\lambda}\right)\theta(z)\,dH^k$$

$$= \underbrace{\frac{1}{\lambda^k}\int_{E-M_i} f\left(\frac{z-x}{\lambda}\right)\theta(z)\,dH^k}_{**} + \underbrace{\frac{1}{\lambda^k}\int_{M_i} f\left(\frac{z-x}{\lambda}\right)\theta(z)\,dH^k}_{***}$$

$$** \le \frac{C_f}{\lambda^k}\int_{(E-M_i)\cap B(x,\lambda R)} \theta(z)\,dH^k = \frac{C_f}{\lambda^k}\,\mu\big(\underbrace{B(x,\lambda R) \cap (E - M_i)}_{****}\big)$$

(but for μ a.e. $x \in M_i$ $\quad **** < \varepsilon\,\alpha(k)(\lambda R)^k$ for any $\varepsilon > 0$ as $\lambda \downarrow 0$)

$$\le \varepsilon\, C_f\, R^k \qquad \varepsilon > 0 \text{ arbitrary}$$

$$\Rightarrow \quad ** \xrightarrow[\lambda \downarrow 0]{} 0 .$$

$$\text{so} \quad \lim_{\lambda \to 0} * = \lim_{\lambda \to 0} *** = \lim_{\lambda \to 0}\left(\frac{1}{\lambda^k}\int_{M_i \cap B(x,\lambda R)} f\left(\frac{z-x}{\lambda}\right)\theta(z)\,dH^k + \underbrace{\frac{1}{\lambda^k}\int_{(N_i - M_i)\cap B(x,\lambda R)} f\left(\frac{z-x}{\lambda}\right)\theta(z)\,dH^k}_{*}\right)$$

Since $H^k((N_i - M_i) \cap B(x,\lambda R)) \le \varepsilon\, H^k(N_i \cap B(x,\lambda R)) \le \varepsilon\,\alpha(k)\,\lambda^k R^k$

Since $* \xrightarrow[\lambda \to 0]{} 0$, using $\dfrac{H^k(M_i \cap B)}{H(N_i \cap B)} \xrightarrow[\lambda \to 0]{} 1$ μ a.e. x in M_i and $\theta^{*k}(H^k, N_i, x) \le 1$ μ a.e. $x \in M_i$

So

$$\lim_{\lambda\downarrow 0}\int_{\eta_{x,\lambda\#}(E)} f(y)\,\theta(x+\lambda y)\,dH^k = \lim_{\lambda\to 0}\theta_i\int_{\eta_{x,\lambda\#}(N_i)} f(y)\,dH^k$$

since $\theta=\theta_i$ on N_i

$$= \lim_{\lambda\downarrow 0}\frac{\theta_i}{\lambda^k}\int_{N_i\cap B(x,\lambda R)} f\left(\tfrac{z-x}{\lambda}\right)dH^k$$

since supp(f) is contained in $B(0,R)$

Since N_i is a C^1 embedded submanifold of $\mathbb{R}^n$, we can choose λ small enough to make $N_i\cap B(x,\lambda R)$ arbitrarily close to $TN_i\cap B(x,\lambda R)$

More precisely:

First since f is continuous on a compact set $\exists\ \delta_{\hat\varepsilon} \ni |f(y)-f(z)|<\hat\varepsilon$ if $|y-z|<\delta_{\hat\varepsilon}$ $\Rightarrow \left|f\left(\tfrac{z-x}{\lambda}\right)-f\left(\tfrac{y-x}{\lambda}\right)\right|<\hat\varepsilon$ if $|y-z|<\delta_{\hat\varepsilon}\lambda$.

Now choose λ small enough and $\phi: U\subset\mathbb{R}^k\to\mathbb{R}^n$ (reparametrizing $\mathbb{R}^n$ if necessary) $\ni$:

$$x=\phi(a)$$
$$\|\phi - I_k^n\|\le\varepsilon$$
$$\|D\phi - I_k^n\|\le\varepsilon$$
$$\|\phi(y)-\phi(a)-D\phi(a)(y-a)\|\le\varepsilon\|y-a\|$$
$$B(x,\lambda R)\cap N_i \subset \phi(B(a,2\lambda R))$$
$$B(x,\lambda R)\cap \widetilde{D\phi(a)}(\mathbb{R}^k)\subset \widetilde{D\phi(a)}(B(a,2\lambda R))$$

where we have chosen ε small enough that $\varepsilon R\le\delta_{\hat\varepsilon}$ and $\varepsilon\le\hat\varepsilon$

$$I_k^n = \begin{array}{cc} & k \\ n & \left[\begin{array}{c} I \\ \hdashline 0 \end{array}\right]\end{array}$$

I is the identity map from $\mathbb{R}^k\to\mathbb{R}^k$ and the above conditions hold on $y\in B(a,2\lambda R)$ ~~[illegible]~~ and $\widetilde{D\phi(a)} = y\to D\phi(a)(y-a)+\phi(a)$. Choose $P_i(x)=D\phi(a)(\mathbb{R}^k)$.

we have that $\frac{1}{\lambda^k}\int_{N_i\cap B(x,\lambda R)} f\left(\frac{z-x}{\lambda}\right) dH^k = \underbrace{\frac{1}{\lambda^k}\int_{B(a,2\lambda R)} f\left(\frac{\phi(y)-\phi(a)}{\lambda}\right) \det(D\phi)\, dH^k}_{*}$

and

$$\frac{1}{\lambda^k}\int_{P_i(a)\cap B(x,\lambda R)} f\left(\frac{z-x}{\lambda}\right) dH^k = \frac{1}{\lambda^k}\int_{B(a,2\lambda R)} f\left(\frac{D\phi(a)(y-a)+\phi(a)-\phi(a)}{\lambda}\right) \det(D\phi(a))\, dH^k$$

$$= \underbrace{\frac{1}{\lambda^k}\int_{B(a,2\lambda R)} f\left(\frac{D\phi(a)(y-a)}{\lambda}\right) \det(D\phi(a))\, dH^k}_{**}$$

so $\quad * - ** = \frac{1}{\lambda^k}\int_{B(a,2\lambda R)} \left(f\left(\frac{\phi(y)-\phi(a)}{\lambda}\right)\det(D\phi) - f\left(\frac{D\phi(a)(y-a)}{\lambda}\right)\det(D\phi(a))\right) dH^k$

$$\leq \frac{1}{\lambda^k}\int_{B(a,2\lambda R)} \hat{\varepsilon}C + \varepsilon C \, dH^k = \frac{\hat{\varepsilon}\, C\, \alpha(k)(2\lambda R)^k}{\lambda^k}$$

$$= \hat{\varepsilon}\, C(k,f)$$

∎

proof of iii) see L. Simon's book pages 62 - 65

End of Lecture: end of UCLA-LANL GMT course! See the GMT course associated with the CDDMA summer school for further developments.

Cheers,

Kevin