Geometric Measure Theory and its Applications

5/7/2007

Addendum 1:

Last week we saw that f was stationary (i.e. $\delta A(f)=0$) when

$$\nabla \cdot \frac{\nabla f}{\sqrt{1+\nabla f \cdot \nabla f}} \qquad (*)$$

and $f : \Omega \to \mathbb{R}$, $\Omega \in \mathbb{R}^2$, Ω convex.

Alternatively, we can think about the graph of f as the zero level set of $F(x,y,z) = f(x,y) - z$. Then $\nabla F = (f_x, f_y, -1)$ and the mean curvature of this 2 dim level set in $\mathbb{R}^3$ is given by

$$\nabla_{x,yz} \cdot \frac{\nabla_{xyz} F}{|\nabla_{xyz} F|} = \nabla_{xyz} \cdot \left(\frac{f_x}{\sqrt{1+\nabla_{xy} f \cdot \nabla_{xy} f}}, \frac{f_y}{\sqrt{1+\nabla_{xy} f \cdot \nabla_{xy} f}}, \underbrace{\frac{-1}{\sqrt{1+\nabla_{xy} f \cdot \nabla_{xy} f}}}_{\text{independent of } z} \right)$$

$$\boxed{\left\{ \begin{matrix} \nabla_{x,yz} \\ \nabla_{xy} \end{matrix} \right\} \left\{ \nabla \text{ in } \begin{matrix} \mathbb{R}^3 \\ \mathbb{R}^2 \end{matrix} \right\}}$$

$$\Rightarrow \; = \nabla_{xy} \cdot \left(\frac{f_x}{\sqrt{1+\nabla_{xy} f \cdot \nabla_{xy} f}}, \frac{f_y}{\sqrt{1+\nabla_{xy} f \cdot \nabla_{xy} f}} \right)$$

$$= (*)$$

So $\delta A(f) = 0 \Rightarrow$ mean curvature of graph is 0.

The above should look very familiar to those of you working with TV regularized functions since:

$$\underbrace{\int |\nabla f|}_{\substack{=\text{ integral over} \\ \text{level set "areas"} \dots \text{ in } \mathbb{R}^2 \text{ this is length of level sets}}} \xrightarrow{\text{Euler-Lagrange}} \nabla \cdot \frac{\nabla f}{|\nabla f|} \Big\} \text{ mean curvature of level sets.}$$

For surfaces which arise as pieces of boundaries of sets of positive reach *, the Steiner-Minkowski formula gives us this same connection between surface area growth and mean curvature.

Steiner-Minkowski

Total mean curvature vector

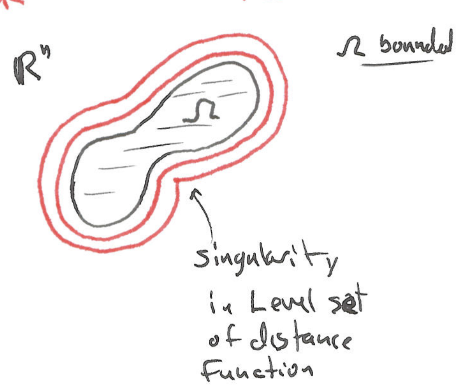
$$\mathcal{H}^K(A+\varepsilon B) = \mathcal{H}^K(A) + \varepsilon\,\mathcal{H}^{K-1}(\partial A) + \frac{\varepsilon^2}{2}\int_{\partial A} \vec{H}\cdot\vec{n}\, d\mathcal{H}^{K-1} + O(\varepsilon^3)$$

This is valid for $\varepsilon \leq r_A$ = reach of A. Note also that $B = B(0,1)$ and $A + \varepsilon_1 B + \varepsilon_2 B = A + (\varepsilon_1+\varepsilon_2)B$. Computing, we get $\frac{d}{d\varepsilon}\left(\mathcal{H}^K(A+\varepsilon B)\right)\Big|_{\varepsilon=0} = \mathcal{H}^{K-1}(\partial A)$. Substituting in $A+\varepsilon B \to A$ we get that $\frac{d}{d\varepsilon}\left(\mathcal{H}^K(A+\varepsilon B)\right) = \mathcal{H}^{K-1}(\partial(A+\varepsilon B))$ (since we have that $\frac{d}{d\varepsilon}\mathcal{H}^K(A+\varepsilon B)\Big|_{\varepsilon=\varepsilon_1} = \frac{d}{d\varepsilon}\mathcal{H}^K(A+\varepsilon_1 B+\varepsilon B)\Big|_{\varepsilon=0}$, $\varepsilon_1 < r_A$)

$$\Rightarrow \quad \frac{d}{d\varepsilon}\mathcal{H}^{K-1}(\partial(A+\varepsilon B))\Big|_{\varepsilon=0} = \frac{d^2}{d\varepsilon^2}\mathcal{H}^K(A+\varepsilon B)\Big|_{\varepsilon=0} = \int_{\partial A} \vec{H}\cdot\vec{n}\, d\mathcal{H}^{K-1}$$

So again, the mean curvature pops up as the First variation of area.

Actually even though the above is valid for sets of positive reach, the conclusion relates mean curvature to the variation of area is very general. For example, Allard showed that this holds for varifolds in his 1972 paper.



* Positive reach: For each $y \in \mathbb{R}^n$, $S_y \equiv \{x \in \bar{\Omega} \mid d(y,x) = d(y,\Omega)\}$. $M \equiv \{y \mid \mathcal{H}^0(S_y) > 1\}$. Define the reach r_Ω by $r_\Omega \equiv d(M,\Omega)$. Ω has positive reach if $r_\Omega > 0$.

Alternatively, r_Ω is the level of the distance function where singularities appear. Define $L_r \equiv \{x \mid d(x,\Omega) = r\}$. Then L_r is nonsingular for $r < r_\Omega$ and r_Ω = infimum of $r \ni L_r$ is singular somewhere. By singular I mean nondifferentiable.

Addendum 2: Calibrations

A calibration is a form ϕ such that

1) $d\phi = 0$ (i.e. it is closed)
2) $\|\phi\| \leq 1$
3) $\langle \eta_T, \phi \rangle = 1$ μ_T a.e. on T, where η_T is the k-vectorfield on T the k-current with measure μ_T.

If we can find such a calibration for T, we are done (in the mission to prove T minimizing). Suppose we have another k-current $S \ni \partial S = \partial T$. This implies that $\partial(S-T) = 0$.

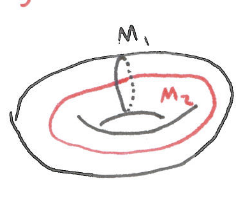
If S & T live in a space with trivial homology classes then $\partial(S-T) = 0 \Rightarrow \exists R \ni S-T = \partial R$. If the space has a nontrivial homology* structure, then what follows is valid only for S and T in the same homology class i.e. $S - T = \partial R$ for some R. $\mathbb{R}^n$ has a trivial homology class structure in all dimensions

Now since we have that $S - T = \partial R$ and $d\phi = 0$

$\Rightarrow S(\phi) - T(\phi) = \partial R(\phi) = R(d\phi) = 0$

$\Rightarrow M(S) \geq S(\phi) = T(\phi) = M(T)$ due to the fact that $\langle \eta_T, \phi \rangle = 1$ μ_T a.e. on T.

* Two submanifolds are in the same homology class if the differ by a boundary ... $M_1 - M_2 = \partial N$. Note that this implies that $\partial(M_1 - M_2) = \partial^2 N = 0$ But that it is easy to construct examples of M_1 & $M_2 \ni \partial(M_1 - M_2) = 0$ but $M_1 - M_2 \neq \partial\{\text{of anything}\}$.



$\partial M_1 = 0$
$\partial M_2 = 0$
so $\partial(M_1 - M_2) = 0$
but $M_1 - M_2 \neq \partial$ of any 2 dim submanifold

Products of currents

Given $T \in \mathcal{D}_n(U)$, $S \in \mathcal{D}_m(V)$ we want to define $T \times S \in \mathcal{D}_{n+m}(U \times V)$. We do this as follows:

If $\omega \in \mathcal{D}^{n+m}(U \times V)$ is represented as

$$\omega = \sum_{|\alpha|+|\beta| = n+m} a_{\alpha\beta}(x,y)\, dx^\alpha \wedge dy^\beta$$

α & β are the usual multindex notation.

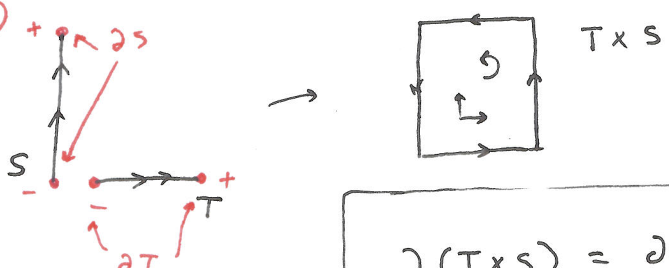
then $$[T \times S](\omega) \equiv T\Big(\sum_\alpha S\big(\sum_\beta a_{\alpha\beta}(x,y)\, dy^\beta\big)\, dx^\alpha\Big)$$

This gives us both

1) $|\alpha| = n$, $|\beta| = m$ for $(T \times S)(dx^\alpha \wedge dy^\beta) \neq 0$

2) $\partial(T \times S) = \partial T \times S + (-1)^n T \times \partial S$

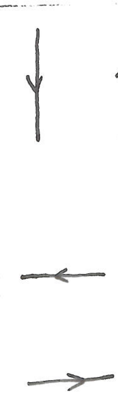
Examples:

① 

$\partial(T \times S) = \partial T \times S$

$+$

$- T \times \partial S$

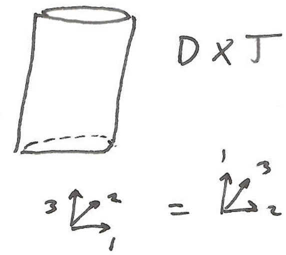


2

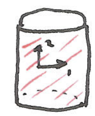
T

D

$D \times T$

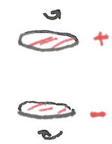


$\partial(D\times T) = \partial D \times T$

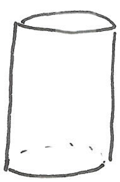


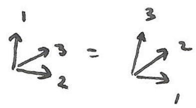
+

$D \times \partial T$

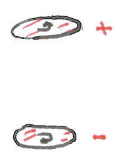


$T \times D$



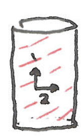


$\partial(T\times D) = \partial T \times D$



+

$(-T\times\partial D)$

(where $T \times \partial D =$ )

Push Forward of a Current (again)

We start with a smooth map $f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ or even simply $f: U \to V$, $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$. We require that f is proper on the support of the current T we want to push forward, i.e. $f^{-1}(K) \cap \operatorname{spt} T$ is compact when K is compact. (why? Because we need $\omega \circ f$ to have compact support in U when ω has compact support in V.)

Recall that the pullback of $\omega \in \mathcal{D}^k(V)$ by f, denoted $f^\#\omega$, is given (pointwise) by:

$$\langle \eta(x), f^\#\omega \rangle = \langle \Lambda^k Df(\eta), \omega(f(x)) \rangle$$

push forward of η by Df i.e. if $\eta = v_1 \wedge v_2 \wedge \dots \wedge v_k$

$\Lambda^k Df(\eta) = Df(v_1) \wedge Df(v_2) \wedge \dots \wedge Df(v_k)$

Now we can define the push forward of T by f

$$f_\# T(\omega) = T(\xi f^\# \omega)$$

smooth cutoff function on U, compactly supported and $\operatorname{spt} T \subset \{x \mid \xi(x) = 1\}$

Basic Properties

1) $\partial f_\# T = f_\# \partial T$

This follows from the fact that $d f^\#\omega = f^\# d\omega$ (verify from definitions). Then

$$\begin{aligned}
\partial f_\# T(\omega) = f_\# T(d\omega) &= T(\xi f^\# d\omega) \\
&= T(\xi d f^\# \omega) \\
&\overset{*}{=} \partial T(\xi f^\# \omega) \\
&= f_\# \partial T(\omega)
\end{aligned}$$

$*$ because $d(\xi f^\#\omega) = \xi d f^\#\omega$ on $\operatorname{spt} T$

2) If T has finite mass and is therefore representable by integration by

$$T(\omega) = \int \langle \vec{T}, \omega \rangle \, d\mu_T$$

we have

$$f_{\#}T(\omega) = \int \langle \vec{T}, f^{\#}\omega \rangle \, d\mu_T$$

*) $$= \int \langle \Lambda^k Df(\vec{T}), \omega(f(x)) \rangle \, d\mu_T$$

3) Using the area formula and the fact that $\Lambda^k Df(\vec{T}) = Jf \dfrac{\Lambda^k Df(\vec{T})}{|\Lambda^k Df(\vec{T})|}$ to see that *) becomes

$$= \int \left\langle \frac{\Lambda^k Df(\vec{T})}{|\Lambda^k Df(\vec{T})|}, \omega(f(x)) \right\rangle Jf \, d\mu_T(x)$$

**) $$= \int \left\langle \sum_{x \in f^{-1}(y)} \frac{\Lambda^k Df(\vec{T}(x))}{|\Lambda^k Df(\vec{T}(x))|}, \omega(y) \right\rangle d\mu_{f(T)}(y)$$

where $\mu_{f(T)} \equiv \mu \circ f^{-1}$ i.e. $\mu_{f(T)}(A) = \mu(f^{-1}(A))$

The most general measures I am sure the area formula is valid for are Hausdorff measures restricted to rectifiable sets.

4) If f is 1-1 we get that *) becomes

$$f_{\#}T(\omega) = \int_{f(\operatorname{spt} T)} \left\langle \frac{\Lambda^k Df(\vec{T}(f^{-1}(y)))}{|\Lambda^k Df(\vec{T}(f^{-1}(y)))|}, \omega(y) \right\rangle d\mu_{f(T)}(y)$$

5) If T, and therefore $f_{\#}T$, are rectifiable then for $\mu_{f(T)}$ a.e. y in $f(\operatorname{spt} T)$, $\displaystyle\sum_{x \in f^{-1}(y)} \frac{\Lambda^k Df(\vec{T}(x))}{|\Lambda^k Df(\vec{T}(x))|} = \tau(y)$ where τ is a unit k vector in $\mathbb{R}^m$ tangent to and orienting $f(\operatorname{spt} T)$ at y.

$\Rightarrow$ **) becomes $f_{\#}T(\omega) = \displaystyle\int \langle \tau(y), \omega(y) \rangle \, d\mu_{f(T)}$