Geometric Measure Theory and its Applications
5/21/2007

Deformation Theorem: Again

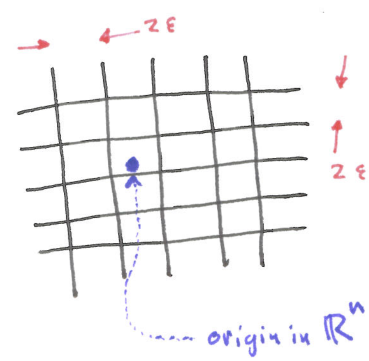
Given $T \in I_m(\mathbb{R}^n)$ and $\varepsilon > 0$ $\exists$ $P \in \mathcal{P}_m(\mathbb{R}^n)$, $Q \in I_m(\mathbb{R}^n)$
$S \in I_{m+1}(\mathbb{R}^n)$ $\ni$
(with $\gamma = 2n^{2m+2}$)

(1)
$$T = P + Q + \partial S$$
$$M(P) \le \gamma\left(M(T) + \varepsilon M(\partial T)\right)$$
$$M(\partial P) \le \gamma M(\partial T)$$
$$M(Q) \le \varepsilon \gamma M(\partial T)$$
$$M(S) \le \varepsilon \gamma M(T)$$

(2) (1) clearly implies that
$$F(T-P) \le \mathcal{F}(T-P) \le \varepsilon\gamma\left(M(T) + M(\partial T)\right)$$

(3) $\operatorname{spt}(P) \subset 2\varepsilon$ grid in $\mathbb{R}^n$

(this some 2ε grid. I.e. take the standard 2ε grid in $\mathbb{R}^n$ that contains the origin. Now translate it by some element of $\mathbb{R}^n$ to get another grid: any such grid is a 2ε grid)

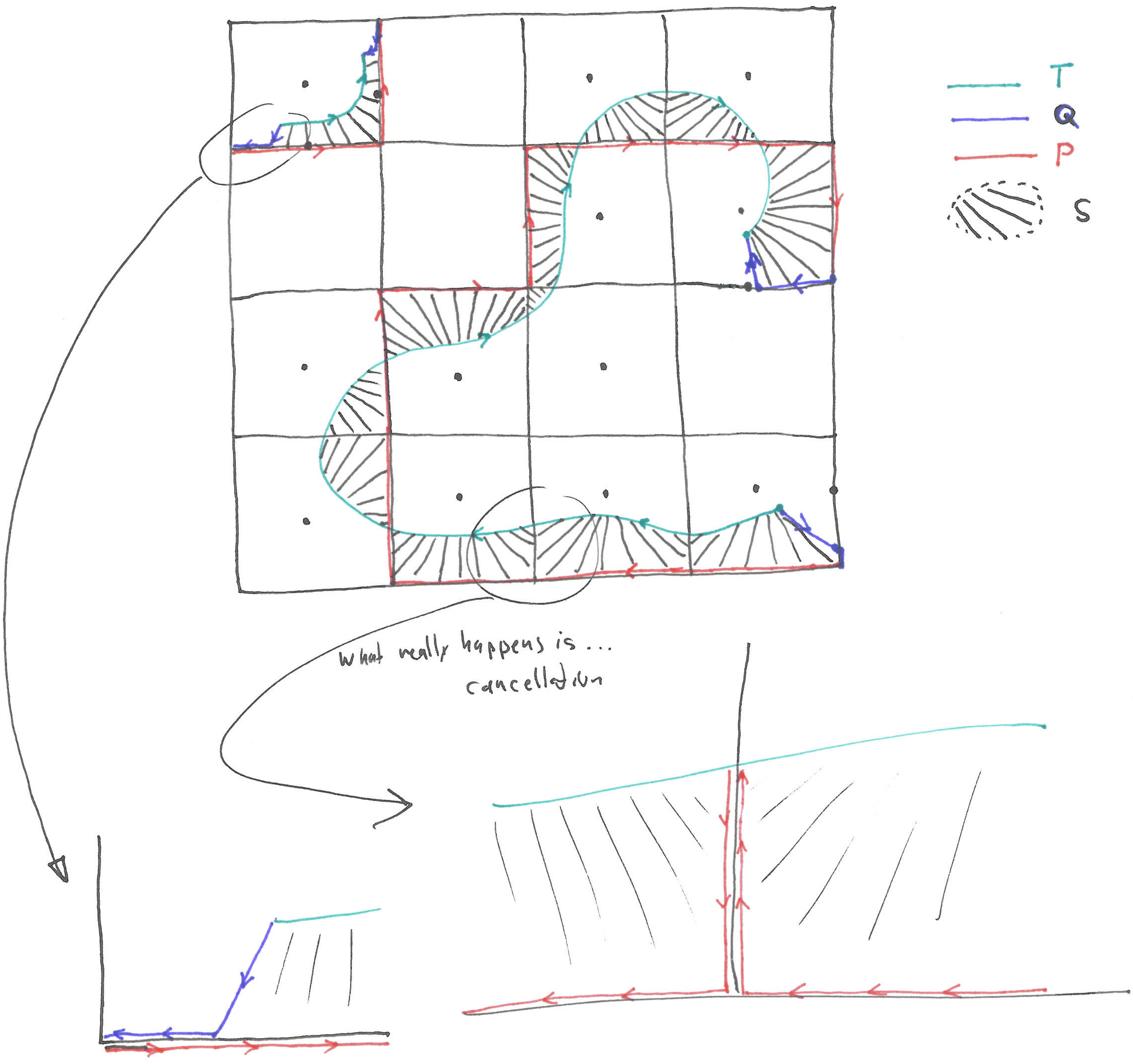


a 2ε grid =
$2\varepsilon\mathbb{Z}^n + x$ (some x in $\mathbb{R}^n$)

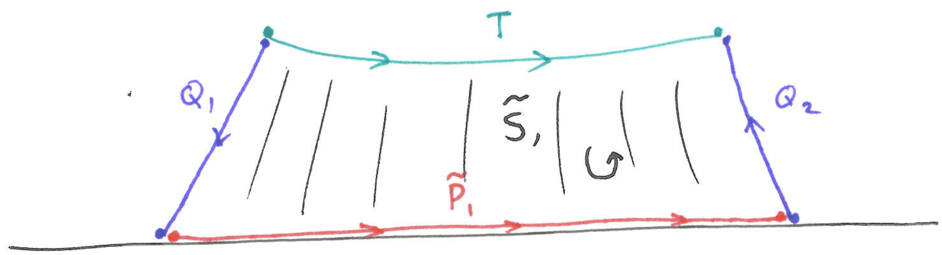
(4) $\operatorname{spt}(P) \cup \operatorname{spt}(Q) \cup \operatorname{spt}(S) \subset \{x \mid \operatorname{dist}(x, \operatorname{spt} T) \le 2n\varepsilon\}$

Now pictures and a few remarks.

After perturbing the grid to avoid as much as possible of the mass of T close to the grid centers, push T from centers to grid...



Q = current swept out by ∂T. This is done until ∂T has been swept onto correct dimensional subgrid. In the picture above this takes 2 sweeps.



let h be the homotopy between the identity and g_1, the map taking T to its retraction (or "push") onto the 2-ε grid.

$$h(t,x) = t\,g_1(x) + (1-t)\,x$$

$$h_\#(0,T) = T$$

$$h_\#(1,T) = g_{1\#}(T)$$

$$h_\#(I \times T) = \tilde{S}_1 \text{ above} \quad (\text{we denote } [0,1] \text{ by } I)$$

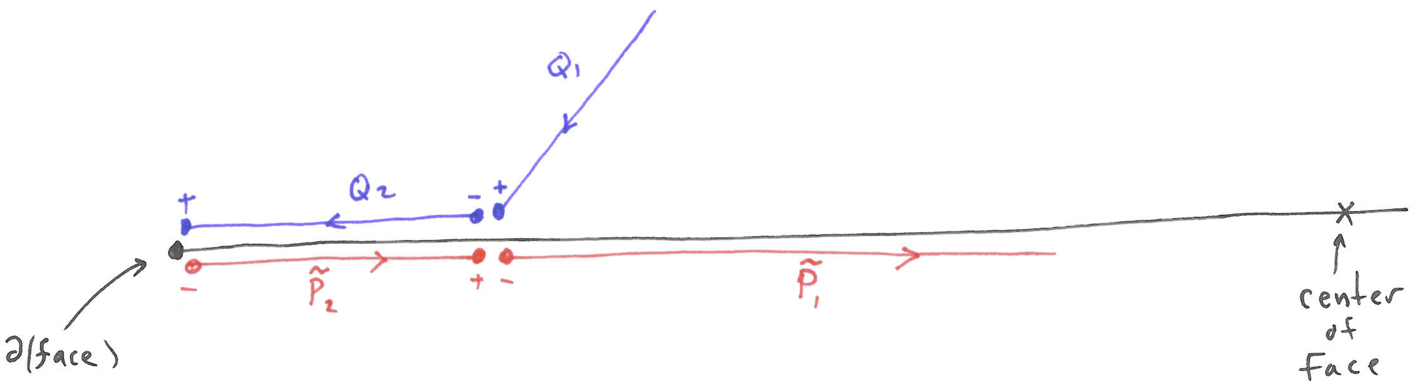
Recalling our brush with the homotopy formula, we have that

$$\partial h_\#(I \times T) = h_\#(\partial(I \times T)) = h_\#(\{1\} \times T - \{0\} \times T - I \times \partial T)$$
$$= g_{1\#}(T) - T - h_\#(I \times \partial T)$$

~~Alaaa~~ Above we see that

$$Q_1 = -h_\#(I \times \{-1\})$$
$$Q_2 = -h_\#(I \times \{+1\})$$
$$\tilde{P}_1 = g_{1\#}(T)$$
$$\tilde{S}_1 = h_\#(I \times T)$$

<u>So</u>: we get everything from $h_\#(I \times T)$ ... the orientations ~~on~~ of the pieces can simply be read off of the orientations on $h_\#(\partial(I \times T))$.



We use the homotopy between the identity and the retraction onto ∂(face) in face $\hat{h}(t,x) = t\,g_2(x) + (1-t)\,x$ to get Q_2 and $\tilde{P}_2$.

Q_2 is the push forward (by $\hat{h}$) of $I \times (\partial Q_1 \cap \text{Face})$.
$\tilde{P}_2$ is the " " " $I \times (\partial \tilde{P}_1$ to the left of center$)$

$$Q_2 = h_\#(I \times \{+1\})$$
$$\tilde{P}_2 = h_\#(I \times \{-1\})$$

Isoperimetric Inequality: an application of the deformation theorem

Theorem If $T \in I_m \mathbb{R}^n$ with $\partial T = 0$ then $\exists$ $S \in I_{m+1}\mathbb{R}^n$ $\ni$ $\partial S = T$ and

$$M(S)^{\frac{m}{m+1}} \leq \gamma\, M(T)$$

where the γ is from the deformation theorem.

Remark: γ is not an optimal constant. In 1986 Fred Almgren proved that this inequality holds for γ = {the constant obtained usg S = $m{+}1$-dim disk and $T = \partial S$} $= \dfrac{1}{n\,\alpha(n)^{1/n}}$

where $\alpha(n)$ = volume of unit ball in $\mathbb{R}^n$.

Proof: choose ε so that $\gamma M(T) = \varepsilon^m$.

Find the decomposition theorem decomposition on the 2-ε grid — $T = P + Q + \partial S$.

Looking at ① in the statement of the deformation theorem, we see that $Q = 0$ (since $\partial T = 0$). Since P lives on the 2-ε grid implies that $M(P) = K(2\varepsilon)^m$ for some $K \in \{\text{non-negative integers}\}$. But ① again tells us that $M(P) \le \gamma M(T) = \varepsilon^m$ $\Rightarrow$ $K = 0$ $\Rightarrow$ $P = 0$. Therefore $T = \partial S$ and ① tells us that $M(S) \le \varepsilon \gamma M(T) = (\gamma M(T))^{\frac{m+1}{m}}$.

$$\Rightarrow \quad (M(S))^{\frac{m}{m+1}} \le \gamma M(T)$$

Compactness:

Theorem Let K be a closed ball in $\mathbb{R}^n$, and $0 \le c < \infty$. Then

$$\mathcal{T}_{K,c} \equiv \{ T \in \mathbf{I}_m(\mathbb{R}^n) : \operatorname{spt} T \subset K,\ M(T) \le c,\ M(\partial T) \le c \}$$

is $\mathcal{F}$ compact.

Remarks: This follows from the closure theorem which says $\mathcal{T}_{K,c}$ is $\mathcal{F}$ complete and the deformation theorem which tells us that $\mathcal{T}_{K,c}$ is totally bounded (under $\mathcal{F}$).

That $\mathcal{T}_{K,c}$ is totally bounded under $\mathcal{F}$ follows from the fact for fixed $\varepsilon > 0$ the number of distinct P's that arise from $T \subset K$ is finite and there is a $P \ni$

$$\mathcal{F}(T-P) \le \varepsilon\gamma\,(M(T) + M(\partial T)) \le 2c\varepsilon\gamma = C(n,m)\,\varepsilon$$

Remarks (cont): The proof of the closure theorem (thm 5.4 in F. Morgan and 4.2.16 in Federer) relies on lots of little details and one difficult result, Federer 4.2.15:

Lemma (4.2.15): If T is a normal current, $\partial T = 0$, and for each $a \in \mathbb{R}^n$, $\partial(T \llcorner B(a,r))$ is rectifiable for almost all $r \in \mathbb{R}^+$ then T is rectifiable.

In 1986 Brian White, using results of Bruce Solomon, found a way to get around the difficult part of the proof of 4.2.15 – the structure theory – to get an alternate, easier proof of the closure theorem.

The Structure theorem [Federer 3.3.13] The highly ingenious, highly technical proof whose was circumvented by White and Solomon states:

Let E be an arbitrary subset of $\mathbb{R}^n$ with $H^m(E) < \infty$. Then E can be decomposed as the union of two disjoint sets $E = A \cup B$ with A (H^m, m) rectifiable and $\mathcal{I}^m(B) = 0$.

Here $\mathcal{I}^m(B)$ is the m-dim integralgeometric measure of B.

Definition of Integralgeometric measure $\mathcal{I}^m$

First $O(n,m)$ is the set of orthogonal projections P mapping $\mathbb{R}^n$ onto some m-plane through the origin.
Next $N(p \mid B, y) = H^0(B \cap P^{-1}(y))$, the number of preimages y has under p, in B.
Finally, $C(n,m)$ will be a normalization constant.

And now the definition:

$$J^m(B) = \frac{1}{C(n,m)} \int_{p \in O(n,m)} \int_{y \in P(\mathbb{R}^n)} N(p|B, y)\, d\mathcal{L}^m_y \, dp$$

$J^m(B)$ is simple the "appropriatly" averaged integral of m-dim areas (counting multiplicity) of projections of B onto all possible m-planes.

"Appropriatly" means it gives

$$J^m(B) = H^m(B)$$

when B is an m-dim rectifiable set. The key to understanding this is that if we average over all $p \in O(n,m)$, the rotation in $\mathbb{R}^n$ of a little flat m-dim piece will not change it's integrated, projected area, and the factor by which this number differs from it's Hausdorff m-dim area is a constant. To fully grasp this think about projections of 2-d surfaces on 2-d planes and satisfy yourself that one can get the appropriate constant by considering the 2-d sphere in $\mathbb{R}^3$ and it's projection (a constant 2-d disk with multiplicity 2) ... this gives $C(3,2)$ very easily.

Existence of Area Minimizing Surfaces

Thm: Let B be an $(m-1)$-dimensional rectifiable current in $\mathbb{R}^n$ with $\partial B = 0$. Then there is an m-dimensional area minimizing current S with $\partial S = B$.

Remark: Since $M(B) < \infty$ both B and S will be integral currents.

Now for the proof.

proof: Since B is rectifiable and $\partial B = 0$, B is an integral $(m-1)$-current. Using the isoperimetric inequality we get $T \in I_{m+1}(\mathbb{R}^n) \ni \partial T = B$ and $M(T) < \infty$. So $M^* \equiv \inf M(W)$ where $W \in I_{m+1}(\mathbb{R}^n)$, $\partial W = B$ is finite.

Choose $S_i \in I_{m+1}(\mathbb{R}^n) \ni \lim M(S_i) = M^*$ requiring of course that $\partial S_i = B \ \forall i$.

The compactness theorem can be applied as long as B and the S_i are all contained in $B(0,R)$ for some $R < \infty$. Choose $R \ni B \subset B(0,R)$. Let P be the radial projection of $\mathbb{R}^n$ onto $\overline{B}(0,R)$. P leaves $\overline{B}(0,R)$ fixed and is distance decreasing and therefore area decreasing.

So $\lim_{i \to \infty} M(P_\# S_i) = M^*$, $\partial P_\# S_i = B$ and $\operatorname{spt}\{B, P_\# S_i\} \subset \overline{B}(0,R)$.

Now apply compactness theorem to get $S \ni \mathcal{F}(S - P_\# S_{i_k}) \to 0$ for some subsequence i_k. (We rename $P_\# S_{i_k} \to T_i$)

We now use ① lower semicontinuity of M w.r.t $\mathcal{F}$ and ② continuity ∂ w.r.t $\mathcal{F}$.

Lower semicontinuity wrt $\mathcal{F}$ of Mass

Suppose $T_i, T \in \mathcal{D}_m$ and $T_i \xrightarrow{\mathcal{F}} T$. Recall that

$$\mathcal{F}(T - T_i) = \{\sup (T - T_i)\phi \mid |\phi| \le 1, |d\phi| \le 1\}$$

and $M(T) = \sup\{T(\phi) \mid |\phi| \le 1\}$.

we have $\lim_{i \to \infty} \{\sup (T - T_i)\phi \mid |\phi| \le 1, |d\phi| \le 1\}$

Choose $\phi_T^\varepsilon \ni T(\phi_T^\varepsilon) \geq M(T) - \varepsilon$

and hence $T\left(\frac{1}{K_\varepsilon}\phi_T^\varepsilon\right) \geq \frac{1}{K_\varepsilon} M(T) - \frac{\varepsilon}{K_\varepsilon}$

where we will choose $K_\varepsilon \equiv \sup |d\phi_T^\varepsilon|$.

we know that $(T - T_i)\frac{1}{K_\varepsilon}\phi_T^\varepsilon \xrightarrow[i\to\infty]{} 0$.

$\Rightarrow \quad T_i\left(\frac{1}{K_\varepsilon} Q_T^\varepsilon\right) \geq \frac{1}{K_\varepsilon} M(T) - \frac{2\varepsilon}{K_\varepsilon}$ for big enough i. $\Rightarrow \quad T_i(Q_T^\varepsilon) \geq M(T) - 2\varepsilon$ for i large enough since $T_i(Q_T^\varepsilon) \leq M(T_i)$

we get

$$\boxed{M(T) \leq \liminf_{i\to\infty} M(T_i)}$$

Continuity of ∂ w.r.t $\mathcal{F}$.

Statement: suppose $\mathcal{F}(S - S_i) \to 0$ then $\mathcal{F}(\partial S - \partial S_i) \to 0$

proof:

$$\mathcal{F}(S - S_i) \to 0$$

$$\Downarrow$$

$$\sup_{\substack{|\phi| \leq 1 \\ |d\phi| \leq 1}} (S - S_i)\phi \xrightarrow[i\to\infty]{} 0$$

And

$$\sup_{\substack{|\phi| \leq 1 \\ |d\phi| \leq 1}} (\partial S - \partial S_i)\phi = \sup_{\substack{|\phi| \leq 1 \\ |d\phi| \leq 1}} (S - S_i)\, d\phi$$

$$\leq \sup_{\substack{|\phi| \leq 1 \\ |d\phi| = 0}} (S - S_i)\phi$$

$$\leq \sup_{\substack{|\phi| \leq 1 \\ |d\phi| \leq 1}} (S - S_i)\phi$$

$$= F(S - S_i) \xrightarrow[i\to\infty]{} 0$$

$$\Rightarrow \quad \mathcal{F}(\partial S - \partial S_i) \xrightarrow[i\to\infty]{} 0$$

Completion of proof of existence of area minimizers:

We have that $\mathcal{F}(S - T_i) \to 0$. By lower semicontinuity of Mass w.r.t $\mathcal{F}$ we have

$$M(S) \leq \liminf_{i \to \infty} M(T_i)$$

$\Rightarrow$ S is area minimizing as long as $\partial S = B$.

But we have that

$$\mathcal{F}(\partial S - \partial T_i) \to 0$$

or

$$\mathcal{F}(\partial S - B) = 0 \quad \Rightarrow \quad \partial S = B.$$

<u>Comment</u>: It is <u>not</u> true that

$$M(S - S_i) \to 0$$

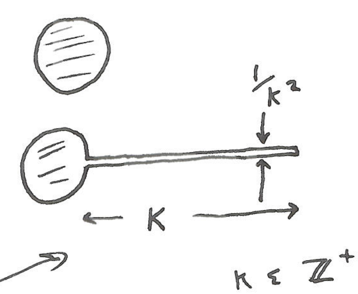
$$\Downarrow$$

$$M(\partial S - \partial S_i) \to 0.$$

<u>Example</u>:

S = unit disk in $\mathbb{R}^2$

S_k = unit disk in $\mathbb{R}^2$ unioned with thin strip which is simultaneously getting longer and skinnier



$$M(S - S_k) = \frac{1}{k} \underset{k \to \infty}{\to} 0$$

$$M(\partial S - \partial S_k) \cong 2k \underset{k \to \infty}{\to} \infty$$