Geometric Measure Theory and its Applications

5/14/2007

Homotopy Formula

We now establish a useful homotopy formula and use it to get two simple but important results (lemmas).

let $h(0,x) = f(x)$, $h(1,x) = g(x)$ for a homotopy mapping $[0,1] \times U \to V$. Then we compute:

$$\partial h_{\#}([\![0,1]\!] \times T) = h_{\#} \partial([\![0,1]\!] \times T)$$

$$= h_{\#}\partial(\{1\} \times T - \{0\} \times T - [\![0,1]\!] \times \partial T)$$

Switching to $I \equiv [\![0,1]\!]$ →

$$= g_{\#}(T) - f_{\#}(T) - h_{\#}(I \times \partial T)$$

The 1-current equals the interval $[0,1]$ with the usual orientation and unit multiplicity.

$\partial [\![0,1]\!] = \{1\} - \{0\}$

$$\Rightarrow \quad g_{\#}(T) - f_{\#}(T) = \partial h_{\#}(I \times T) + h_{\#}(I \times \partial T) \quad (*)$$

Now choose $h(t,x) = t\,g(x) + (1-t) f(x) = f(x) + t\,(g(x) - f(x))$

We next compute a bound on the right hand terms of $(*)$ above.

$$h_\#(I \times T)\,\omega = \int_{\operatorname{spt} T} \langle Dh_\# e_1 \wedge \vec{T}, \omega(h(t,x)) \rangle \, d\mu_T \, dt$$

$$= \int_{\operatorname{spt} T} \Big\langle (g(x) - f(x)) \wedge \big([t\,Dg + (1-t)Df]_\# \vec{T}\big), \omega(h(t,x)) \Big\rangle \, d\mu_T \, dt$$

$$\le \sup_{\operatorname{spt} T} |g(x) - f(x)| \cdot \sup_{\operatorname{spt} T} (\|Dg\| + \|Df\|)\, M(T)\, \|\omega\|$$

$$\Rightarrow \quad M(h_\#(I \times T)) \le \sup_{\operatorname{spt} T} |g(x) - f(x)| \cdot \sup_{\operatorname{spt} T} (\|Dg\| + \|Df\|)\, M(T) \qquad (**)$$

(**) Permits us to bound both terms on the RHS of (*) above and we apply that now to get the two lemmas.

Applications of Homotopy

Lemma 1

If $T \in \mathcal{D}_n(U)$, $M(T), M(\partial T) < \infty$ and $f, g : U \to V$ with $f|_{\operatorname{spt} T} = g|_{\operatorname{spt} T}$ proper Then $f_\# T = g_\# T$

proof:

$h(t,x) = t\,g(x) + (1-t) f(x)$ which as computed above gives

$$g_\# T(\omega) - f_\# T(\omega) = \partial h_\#(I \times T)(\omega) + h_\#(I \times \partial T)(\omega)$$
$$= h_\#(I \times T)(d\omega) + h_\#(I \times \partial T)(\omega)$$

$$\Rightarrow |g_\# T(\omega) - f_\# T(\omega)| \le \sup_{\operatorname{spt} T} |g - f| \cdot \sup \big| \|Df\| + \|Dg\| \big| \cdot M(T)\, |d\omega|$$
$$+ \sup_{\operatorname{spt} T} |g - f| \cdot \sup \big| \|Df\| + \|Dg\| \big| \cdot M(\partial T)\, |\omega|$$

Both $= 0$

$$= 0 \quad \Rightarrow \quad g_\# T = f_\# T$$

lemma 2

$f_\#$ for f Lipschitz: $T \in \mathcal{D}_n(U)$, $M(T), M(\partial T) < \infty$

$f: U \to V$, $f|_{\operatorname{spt} T}$ proper, f Lipschitz, $f^\sigma = h_\sigma * f$

where h_σ is a smooth mollifier with compact support at $x=0$ with diameter $= \sigma$, Then $\lim\limits_{\sigma \to 0} f^\sigma_\#(T)(\omega)$ exists for each $\omega \in \mathcal{D}^n(V)$ and we define $f_\#(T)(\omega)$ to be this limit.

(additionally $\operatorname{spt} f_\#(T) \subset f(\operatorname{spt} T)$ and $M(f_\# T) \le (\operatorname{Lip} f)^n M(T)$)

proof:

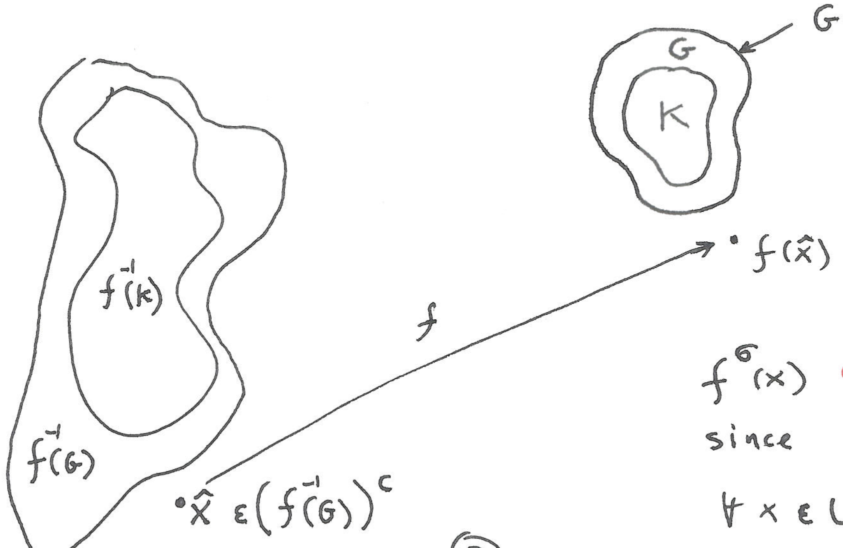
We want to apply (*) from the Homotopy formula above and as always, we need proper mappings. So we need f^σ to be proper if f is:

Since both f and f^σ are continuous, we know that $K \subset V$ compact will be pulled back to closed sets. Since f is proper $f^{-1}(K)$ is compact. We choose $\delta > 0$ big enough that

$$\sup_{x \in U} |f^\sigma(x) - f(x)| < \delta .$$

We can do this because the spt of h_σ has a diameter of $\sigma < \infty$ and f is Lipschitz.

closed $G = \delta$-neighborhood of K



$f^\sigma(x)$ cannot be in K since $|f^\sigma(x) - f(x)| < \delta$ $\forall x \in U$

$$\Rightarrow (f^{\sigma})^{-1}(K) \subset f^{-1}(G) \Rightarrow f^{\sigma} \text{ is proper}$$

Now apply (*) from above to get:

$$\left| f^{\sigma}_{\#}(T)(\omega) - f^{\gamma}_{\#}(T)(\omega) \right| \leq$$

$$\left| h_{\#}(I \times T)(d\omega) \right| + \left| h_{\#}(I \times \partial T)(\omega) \right|$$

$$\leq \sup |f^{\sigma} - f^{\gamma}| \cdot \sup (\|Df^{\sigma}\| + \|Df^{\gamma}\|) \cdot M(T) \cdot \|d\omega\|$$

$$+ \sup |f^{\sigma} - f^{\gamma}| \cdot \sup (\|Df^{\sigma}\| + \|Df^{\gamma}\|) \cdot M(\partial T) \cdot \|\omega\|$$

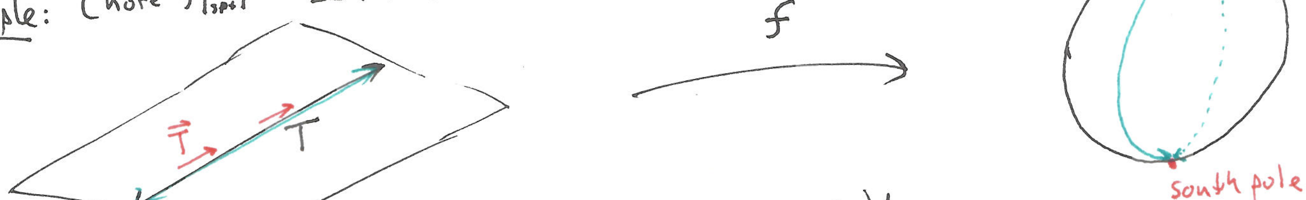
$$= C_{\omega, T, f} \cdot \sup |f^{\sigma} - f^{\gamma}| \longrightarrow 0 \quad \sigma, \gamma \to 0$$

(all sup's taken over $\operatorname{spt} T$)

so $f^{\sigma}_{\#}(T)(\omega)$ is cauchy as $\sigma \to 0$. After obs[erv]ing the linearity and boundedness w.r.t. ω, we <u>define</u> $f_{\#}(T)(\omega)$ to be this limit.

Now we want to establish that $\operatorname{spt} f_{\#} T \subset f(\operatorname{spt} T)$. This requires us to use the fact that $f|_{\operatorname{spt} T}$ is proper.

<u>Example</u>: (note $f|_{\operatorname{spt} T}$ is <u>not</u> proper)



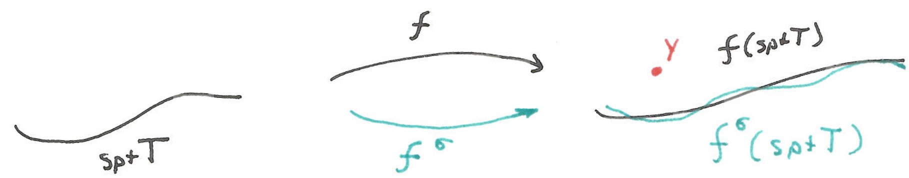
$T =$ entire x-axis, oriented by $\vec{T}$, in $\mathbb{R}^2$

$\operatorname{spt} T =$ entire x-axis

$f(\operatorname{spt} T) =$ great circle on the 2 dim sphere in $\mathbb{R}^3$, $\boxed{\text{missing}}$ the south pole

$\operatorname{spt}(f_{\#} T) =$ the great circle in its entirety
$= f(\operatorname{spt} T) +$ south pole (... therefore, $f|_{\operatorname{spt}}$ must be proper)

(Back to showing $\operatorname{spt} f_\# T \subset f(\operatorname{spt} T)$)



Pick $y \notin f(\operatorname{spt}T)$. Define $\delta = \operatorname{dist}(y, f(\operatorname{spt}T))$. Suppose $\delta > 0$.

Then: $\quad B(y, \frac{\delta}{2}) \cap f^{\sigma}(\operatorname{spt}T) = \phi$ for all $\sigma < \hat{\sigma}$, $\hat{\sigma}$ small enough

$\Rightarrow \quad (f^{\sigma})^{-1}(B(y,\frac{\delta}{2})) \cap \operatorname{spt}T = \phi \quad$ for $\sigma < \hat{\sigma}$

$\Rightarrow \quad y \notin \operatorname{spt} f^{\sigma}_{\#}(T)$, $\quad \sigma < \hat{\sigma} \;\Rightarrow\; y \notin \operatorname{spt} f_{\#}(T)$

Now consider the case that $\delta = 0$. Then $\exists \{y_i\} \subset f(\operatorname{spt}T) \ni \{y_i\} \subset \overline{B(y,1)}$ and $y_i \xrightarrow[i\to\infty]{} y$. Since $f|_{\operatorname{spt}T}$ is proper we have that $f^{-1}(\overline{B(y,1)}) \cap \operatorname{spt}T$ is compact and contains a sequence $x_i \in f^{-1}(y_i) \cap \operatorname{spt}T$. We can therefore find $\hat{x} \in \operatorname{spt}T$ and a subsequence $x_{i_k} \ni x_{i_k} \to \hat{x}$. We obtain that $f(\hat{x}) = y \Rightarrow y \in f(\operatorname{spt}T)$. ∎

Finally, since smoothing (or mollification) does not increase the Lipschitz constant, we get:

$$M(f_\# T) = \sup_{\substack{\omega \in \mathcal{D}^n(V) \\ |\omega| \le 1}} \lim_{\sigma \to 0} \int \langle \Lambda_n Df^{\sigma} \cdot \vec{T}, \omega(f^{\sigma}) \rangle \, d\mu_T$$

$$= \sup_{\substack{\omega \in \mathcal{D}^n(V) \\ |\omega| \le 1}} \lim_{\sigma \to 0} \int \langle \vec{T}, \underbrace{\omega(f^{\sigma}) \circ \Lambda_n Df^{\sigma}}_{\substack{= \gamma \in \mathcal{D}^n(U) \\ |\gamma| \le |\Lambda_n Df^{\sigma}|}} \rangle \, d\mu_T$$

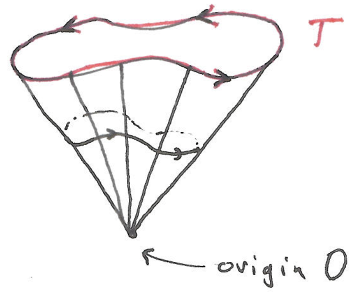
$$\le M(T) \cdot \sup|\omega| \cdot \sup|\Lambda_n Df^{\sigma}|$$

$$\le M(T) \cdot (\operatorname{Lip} f)^n$$

Cones

$$0 \times\!\!\!\!\times T \equiv h_\#(I \times T) \quad \text{with} \quad h(t,x) = tx$$

$\partial(0 \times\!\!\!\!\times T) =$



$$h_\#(\{1\} \times T - \{0\} T)$$
$$- h_\#(I \times \partial T)$$
$$= T - h_\#(I \times \partial T)$$

On to the deformation theorem

In this lecture we simply state the theorem and draw one illustrating picture. In the next lecture we draw more pictures, give key ideas of proof, state and prove some simple but important applications/corollaries.

Deformation Theorem (F. Morgan's Book chap 5, Federer 4.2.9, L. Simon's book section §29)

Given $T \in I_m(\mathbb{R}^n)$ and $\varepsilon > 0$, $\exists$ $P \in \mathcal{P}_m(\mathbb{R}^n)$, $Q \in I_m(\mathbb{R}^n)$ and $S \in I_{m+1}(\mathbb{R}^n)$ $\ni$ for $\gamma = 2n^{2m+2}$

(1) $\quad T = P + Q + \partial S$

(2) $\quad M(P) \leq \gamma(M(T) + \varepsilon M(\partial T))$

$\quad M(\partial P) \leq \gamma(M(\partial T))$

$\quad M(Q) \leq \varepsilon \gamma M(\partial T)$

$\quad M(S) \leq \varepsilon \gamma M(T)$

$\Rightarrow \quad \mathcal{F}(T-P) \leq \varepsilon\gamma(M(T) + M(\partial T))$

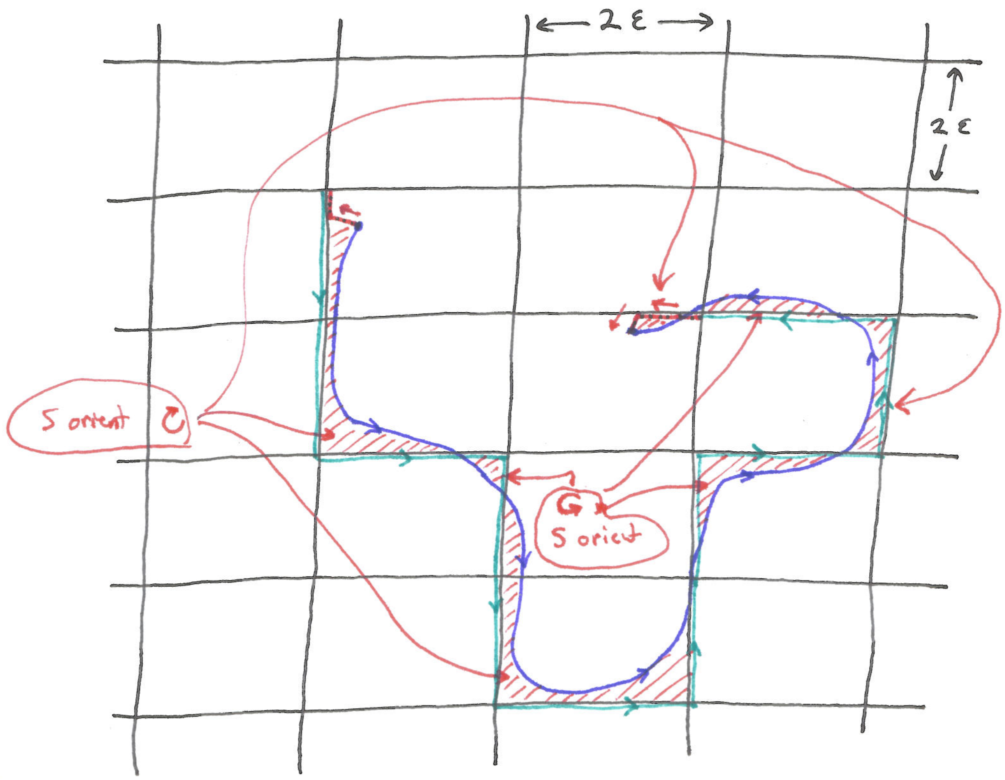
(3) spt P is contained in the m-dimensional 2ε grid. In other words, $x \in$ spt P then at least $n-m$ of its coordinates are even multiples of ε.

spt ∂P is contained in the $m-1$-dimensional 2ε grid

(4) $$\text{spt } P \cup \text{spt } Q \cup \text{spt } S \subset \{x \mid \text{dist}(x, \text{spt } T) \le 2n\varepsilon\}$$

A picture

T is a 1-current in $\mathbb{R}^2$.



T — (Blue)

P — (Green)

Q — (red with black dots)

S — red hatching