Geometric Measure Theory and its Applications

4/30/2007

In this lecture we jump back to facts about Lipschitz mappings and ahead to a peek at minimal surfaces and isoperimetric inequalities in positive codimension.

Reminder: $f: \mathbb{R}^n \to \mathbb{R}$ is Lipschitz if $\exists$ a constant $C \ni$

$$|f(x) - f(y)| \leq C|x-y| \quad \forall\, x, y \in \mathbb{R}^n$$

(actually, this concept makes sense for mappings from one metric space to another. We will often consider the case that $f: \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz.)

Another look at rectifiability and rectifiable currents

Definition 1: $E \subset \mathbb{R}^n$ is m-countably rectifiable if

$$E = E_0 \bigcup_{i=1}^{\infty} f_i(E_i)$$

where $f_i: \mathbb{R}^m \to \mathbb{R}^n$ are Lipschitz, $m \leq n$, $E_i \subset \mathbb{R}^m$, E_0 has $\mathcal{H}^m$ measure 0.

Another equivalent definition is given by:

Definition 2: $E \subset \mathbb{R}^n$ is m-countably rectifiable if

$$E = E_0 \bigcup_{i=1}^{\infty} E_i$$

where $E_i \subset M_i$ a C^1 m-dim submanifold of $\mathbb{R}^n$ and $\mathcal{H}^m(E_0) = 0$.

Another definition that can be thought of as almost a definition of rectifible set is the definition of rectifiable current given by Federer in 4.1.24

Definition 3: We $T \in \mathcal{D}_m(U)$, $U \subset \mathbb{R}^n$ is a rectifiable current with support in $K \subset\subset U$ if there is an m-dim polyhedral current $P_\varepsilon \subset \mathbb{R}^k$ and a Lipschitz map $f_\varepsilon : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$M(T - f_{\varepsilon\#} P_\varepsilon) < \varepsilon$$

For any $\varepsilon > 0$.

What is $f_\#$?: Answer - the push forward by f. What is that?

Answer:

1) $f_\# T$ for a general $T \in \mathcal{D}_m$ is defined to be

$$f_\# T(\omega) \equiv T(f^\# \omega)$$

where $f : \mathbb{R}^k \to \mathbb{R}^n$, ω is an m-form in $\mathbb{R}^n$ and $f^\# \omega$ is the pullback of ω to $f^\# \omega$, an m-form in $\mathbb{R}^k$. The pullback of ω $f^\# \omega$ is defined by it's action on m-vector fields in $\mathbb{R}^k$, $\xi(x)$

$$\langle \xi(x), f^\# \omega \rangle \equiv \langle \Lambda_m Df(\xi), \omega(f(x)) \rangle.$$

Finally $\Lambda_m Df(\xi)$ is the linear map on m-vectors obtained by mapping

$$v_1 \wedge v_2 \cdots \wedge v_m \longrightarrow Df(v_1) \wedge Df(v_2) \cdots \wedge Df(v_m)$$

2) What does 1 boil down to? For rectifiable m-currents T, $f_\# T$ is the current whose support

$\mathrm{supp}(f_\# T) = f(\mathrm{supp}\, T)$ (modulo comment) and

$$\overrightarrow{f_\# T} = \frac{(\Lambda_m Df)(\vec{T})}{|(\Lambda_m Df)(\vec{T})|},$$ the normalized m-vector

obtained by mapping the m-vector of $\vec{T}$.

comment: if f is not one to one then

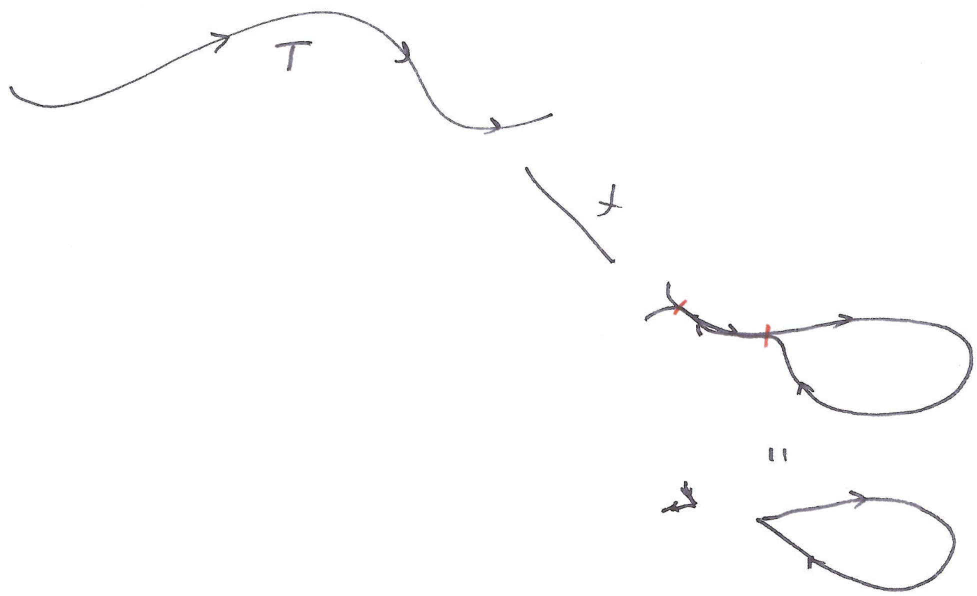
$$\overrightarrow{f_\# T}(y) = \sum_{f^{-1}(y)} \frac{(\Lambda_m Df)(\vec{T})}{|(\Lambda_m Df)(\vec{T})|}$$

and

$\operatorname{supp} f_\# T$ might be a strict subset of $f(\operatorname{supp} T)$

due to cancellation

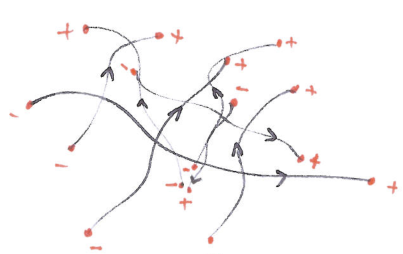
Example: cancellation



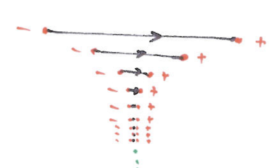
oh yeah: $d\|T\| = \mathcal{H}^m \llcorner \operatorname{supp} T$

$d\|f_\# T\| = \mathcal{H}^m \llcorner \operatorname{supp}(f_\# T)$

Examples of Rectifiable sets and currents:

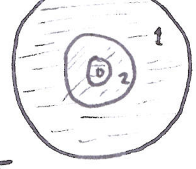
1) 

Rect. 1-current

2) 

Rect. 1-current

3)



T

$r_1 > r_2 > r_3 \cdots$

$r_i \to 0$

Add disks of radius r_i in $\mathbb{R}^2$, each centered at the origin. we get a rectifiable current whose boundary is the union of circles with radius's r_i.

$$M(T) = \sum_{i=1}^{\infty} \pi r_i^2$$

$$M(\partial T) = \sum_{i=1}^{\infty} 2\pi r_i$$

choosing the r_i's to be $\frac{1}{i}$ we get a current with finite mass whose boundary has infinite mass. So T is rectifiable but not integral.

Properties of Lipschitz Maps

- **Theorem (Rademacher):** If $f: \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, then f is differentiable almost everywhere.

- **Theorem (Approximation):** If $f: \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, then for every $\varepsilon > 0$ $\exists$ $f_\varepsilon \in C^1(\mathbb{R}^n)$ $\ni$

$$\mathcal{L}^n(\{x \mid f \neq f_\varepsilon\} \cup \{x \mid \nabla f(x) \neq \nabla f_\varepsilon(x)\}) < \varepsilon$$

- **Theorem (Extension):** If $f: A \subset \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, then $\exists$ $\bar{f}: \mathbb{R}^n \to \mathbb{R}$ $\ni$ $Lip(\bar{f}) = Lip(f)$ and $f = \bar{f}|_A$

Such extensions are far from unique and one finds a more interesting optimal extension in "absolutely minimizing functions". A beautiful article on these functions by Aronsson, Crandall & Juutinen appeared in the Bulletin of the AMS. (Vol 41, # 4, p 439-505)

Very briefly: $f: \Omega \to \mathbb{R}$ is an absolutely minimizing function if f continuously extends boundary data without increasing the Lipschitz constant - $Lip_\Omega(f) = Lip_{\partial\Omega}(f)$ and for any $V \subset\subset \Omega$ $Lip_V(f) = Lip_{\partial V}(f)$.

Continuing

■ Theorem (C^1 Sard type theorem)

Suppose $f: M \subset \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^N$, M is an n-dim C^1 submanifold, and f is C^1. Then for $\mathcal{L}^N$- a.e. $y \in f(M)$, $f^{-1}(y)$ decomposes into an $(n-N)$-dim C^1 submanifold and a closed set of $\mathcal{H}^{n-N}$ measure zero.

Recall: If f and M are both C^{n-N+1} then Sard's theorem says that for $\mathcal{L}^n$ a.e. $y \in f(M)$, $f^{-1}(y)$ is an $(n-N)$-dim C^{n-N+1} submanifold.

Area and Coarea Formulas

We next consider how integration over rectifiable sets transforms under Lipschitz maps. This results in the Area and Coarea formulas.

Jacobian, appropriately generalized: for $f: \mathbb{R}^n \to \mathbb{R}^n$ $Jf \equiv \det(Df)$. If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^N$ $n \leq N$ we define $Jf \equiv \sqrt{\det(Df^* \circ Df)}$ where Df^* is the adjoint (transpose) of Df. If $f: \mathbb{R}^n \to \mathbb{R}^k$, $n \geq k$ $Jf \equiv \sqrt{\det(Df \circ Df^*)}$. In the first two cases, Jf gives the dilation f induces on little pieces of $\mathbb{R}^n$ as it maps them forward. In the last case ($f: \mathbb{R}^n \to \mathbb{R}^k$ $n > k$) Jf gives the ~~exact~~ dilation that f induces on the k-dim subspaces orthogonal to the level sets (which have dim $= n-1$) of f. Finally: $J_M f$ will be the above Jacobians in which we use the diff restricted to some manifold or rectifiable set M, $Df|_M$ to compute the Jacobian.

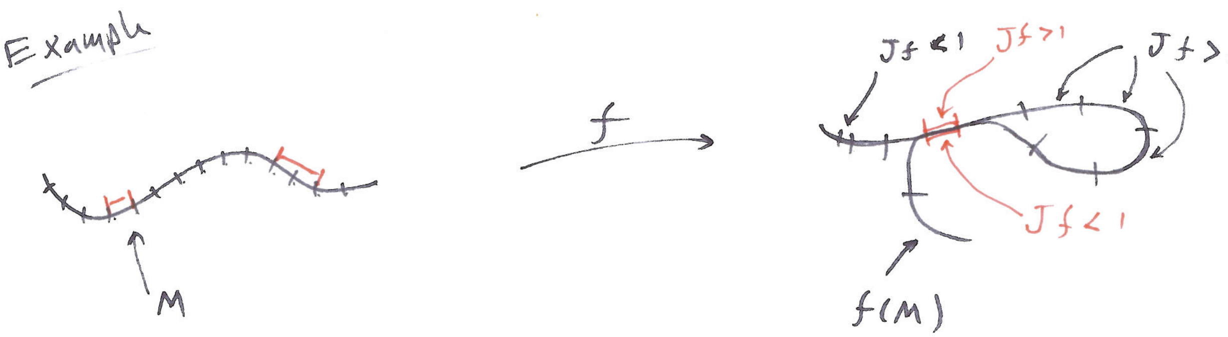
Area Formula: Let $f: M \to \mathbb{R}^N$, f Lipschitz, M m-dim rect $\subset \mathbb{R}^n$ and $m \leq N$. Then

$$\int_M J_M f \, d\mathcal{H}^m = \int_{\mathbb{R}^N} \mathcal{H}^0(f^{-1}(y)) \, d\mathcal{H}^m(y)$$

and, more generally for g $\mathcal{H}^m$ intergrable, $g: M \to \mathbb{R}$

$$\int_M g \, J_M f \, d\mathcal{H}^m = \int_{\mathbb{R}^N} \Big(\sum_{x \in f^{-1}(y)} g(x) \Big) \, d\mathcal{H}^m(y)$$

Example



$f: M \to \mathbb{R}^2$, M 1-rectifiable

If we integrate $\mathcal{H}^1$ over $f(M)$ and correct with a multiplicity equal to the number of preimages a point in $f(M)$ has, we get the $\mathcal{H}^1$ integral over M weighted by the dilation imposed by f.

Coarea Formula :

Let $f: M \longrightarrow \mathbb{R}^k$, f Lipschitz, M m-dim rect $\subset \mathbb{R}^n$ and $m > k$. Then

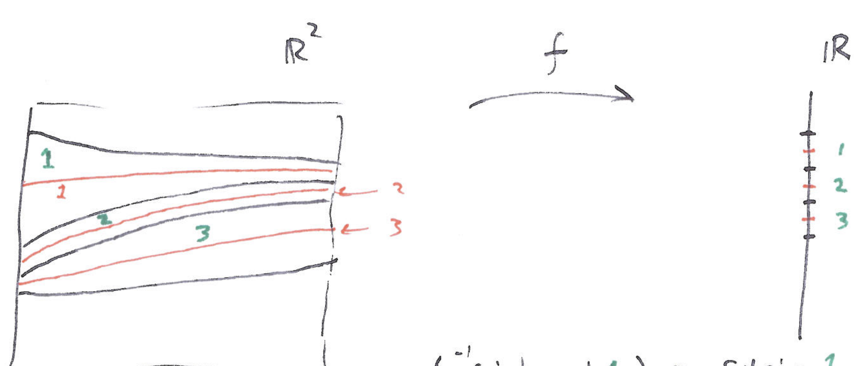
$$\int_M J_M f \, d\mathcal{H}^m = \int_{\mathbb{R}^k} \left(\mathcal{H}^{m-k}(f^{-1}(y))\right) d\mathcal{H}^k(y)$$

more generally if g is $\mathcal{H}^m$ ~~measureable~~ integrable on M, we get

$$\int_M g \, J_M f \, d\mathcal{H}^m = \int_{\mathbb{R}^k} \left(\int_{f^{-1}(x)} g \, d\mathcal{H}^{m-k} \right) d\mathcal{H}^k(x)$$

Example

Jf small at left end of strip 1, and bigger than 1 at the other (right) end.



$\int_{\text{strip 1}} Jf \, d\mathcal{H}^2 \cong$ length(interval 1) $\cdot$ length(curve 1)

f^{-1}(interval 1) = strip 1 etc

Brief Mention of Marstrand and Preiss Theorems

Define: $$\theta^\alpha(\mu, x) \equiv \lim_{r \to 0} \frac{\mu(B(x,r))}{\omega(\alpha)\, r^\alpha}$$ when this limit exists where $\omega(\alpha)$ = volume of unit α-dim ball.

Marstrand (1964) Let μ be a Radon measure and suppose $\theta^\alpha(\mu, x)$ exists and

$$0 < \theta^\alpha(\mu, x) < \infty \quad \text{a.e. } x \in E$$

$\mu(E) > 0$ for some nonnegative α. Then α is an integer.

Proof: Mattila's book chap. 14, Camillo De Lellis notes Chap. 3, Jonas Azzam's talks (from Mattila's book) UCLA analysis seminar - 5/1/2007, 5/8/2007, ?

Preiss (1987)

Suppose μ satisfies the hypotheses of Marstrand's theorem and $\alpha > 0$. Then not only is $\alpha = n \in \mathbb{Z}^+$, $\mu = \mathcal{H}^n \llcorner M$, where M is a n-countably rectifiable set.

Proof: De Lellis notes or Preiss original 1987 paper. De Lellis's notes are on the GMT course webpage.

Now we move ahead, to brief glances at minimal surfaces and the isoperimetric inequality in positive codimension.

First the isoperimetric inequality comments

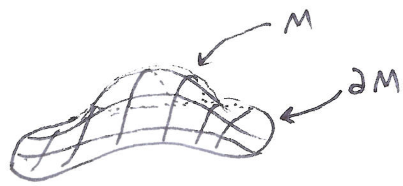
The usual isoperimetric inequality that most of us see first states that for a set $\Omega \subset \mathbb{R}^n$

$$\left(\mathrm{Vol}_n(\Omega)\right)^{\frac{n-1}{n}} \leq C \; \mathrm{Vol}_{n-1}(\partial\Omega)$$

and that $C = \frac{1}{n}\frac{1}{\omega(n)^{1/n}}$, $\omega(n)$ = volume of unit ball in $\mathbb{R}^n$ is optimal. (It is attained for Ω = Ball).

Such Ω are naturally top (i.e. n) dimensional currents in $\mathbb{R}^n$. What can we say about the case in which Ω is a set with positive codimension and we use the Hausdorff measure for which Ω has positive and finite volume?

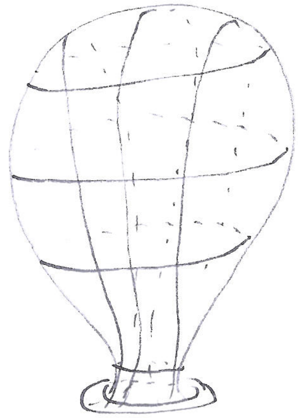
Example



... a little bit of thought is enough to convince us that

$$\left(\mathcal{H}^k(M)\right)^{\frac{k-1}{k}} \not\leq C \left(\mathcal{H}^{k-1}(\partial M)\right)$$

no matter what C we choose. This is due to the fact that M can balloon out ...



What works instead

$$\left(\mathcal{H}^k(M)\right)^{\frac{k-1}{k}} \leq C \left(\mathcal{H}^{k-1}(\partial M) + \int_M \|\vec{H}\| \, d\mathcal{H}^k \right)$$

where $\vec{H}$ is the total mean curvature of M,

A Brief look at minimal surfaces

Theorem: If f is a C^2 real valued function on a planar domain D such that the graph of f is area minimizing, f satisfies

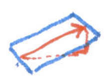
$$(1+f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1+f_x^2)f_{yy} = 0$$

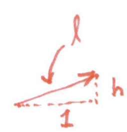
Convesely, if f satisfies this equation on a convex domain then its graph is area minimizing.

By "area minimizing" we mean any other surface having the same boundary has at least as much surface area.

Area of a graph

The area of a little patch of graph above a small square in the domain of the graph is bigger by a factor of $\sqrt{1+f_x^2+f_y^2}$:





$$h = \sqrt{f_x^2+f_y^2}$$

$$|\ell| = \sqrt{1+f_x^2+f_y^2}$$

$\Rightarrow$ Area of graph of f over D, G_f^D, is given by

$$A(G_f^D) = \int_D \sqrt{1+f_x^2+f_y^2}\, dx\,dy .$$

1st Variation of A

$$\delta A = \int \overbrace{\sqrt{1+(f_x+\delta f_x)^2+(f_y+\delta f_y)^2}}^{I} - \sqrt{1+f_x^2+f_y^2}\; dx\,dy$$

we manipulate I :

$$I = \left(\overbrace{\sqrt{1+(f_x+\delta f_x)^2+(f_y+\delta f_y)^2}}^{I_1} - \overbrace{\sqrt{1+f_x^2+f_y^2}}^{I_2} \right) \cdot \frac{(I_1 + I_2)}{(I_1 + I_2)}$$

$$= \frac{2 f_x \delta f_x + (\delta f_x)^2 + 2 f_y \delta f_y + (\delta f_y)^2}{2\sqrt{1+f_x^2+f_y^2}}$$

← here we have let $\delta f \to 0$ ... a bit premature but ok.

$$= \frac{f_x \delta f_x + f_y \delta f_y}{\sqrt{1+f_x^2+f_y^2}}$$

← Keeping only first order terms

$$= \frac{\partial}{\partial x}\left(\frac{f_x}{\sqrt{1+f_x^2+f_y^2}}\right)\delta f + \frac{\partial}{\partial y}\left(\frac{f_y}{\sqrt{1+f_x^2+f_y^2}}\right)\delta f$$

← integration by parts, remembering we are fixing the boundary values and that $A = \int I$

so $$\delta A = \int \nabla \cdot \left(\frac{\nabla f}{\sqrt{1+f_x^2+f_y^2}}\right) \delta f \, dx\,dy$$

$$\Rightarrow \delta A = 0 \Rightarrow \nabla \cdot \frac{\nabla f}{\sqrt{1+f_x^2+f_y^2}} = 0$$

computing this we get and simplifying we get

$$\frac{(1+f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1+f_x^2) f_{yy}}{(1+f_x^2+f_y^2)^{3/2}} = 0$$

$$\Rightarrow (1+f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1+f_x^2) f_{yy} = 0 \qquad (*)$$

To get the other direction (if $(*)$ holds f's graph minimizes) we introduce a calibration.

we want to consider the graph of f over D to be a 2 current, G.

$(e_1 + f_x e_3) \wedge (e_2 + f_y e_3) = e_1 \wedge e_2 + f_y e_1 \wedge e_3 + f_x e_3 \wedge e_2$

orients this surface. Normalizing we get

$$\vec{G} = \frac{1}{\sqrt{1+f_x^2+f_y^2}} \left(e_1 \wedge e_2 + f_y e_1 \wedge e_3 + f_x e_3 \wedge e_2 \right)$$

So our current is

$$G = (\underset{\text{set}}{G_f^D}, \underset{\text{2-vector field}}{\vec{G}}, \underset{\text{measure}}{\mathcal{H}^2 \llcorner G_f^D})$$

This is the 'calibrating form'

Define the 2-form ϕ over $D \times \mathbb{R}$ by

$$\phi(x,y,z) = \frac{-f_x \, dy \wedge dz - f_y \, dz \wedge dx + dx \wedge dy}{\sqrt{f_x^2 + f_y^2 + 1}}$$

this is what makes ϕ a "calibrating form"

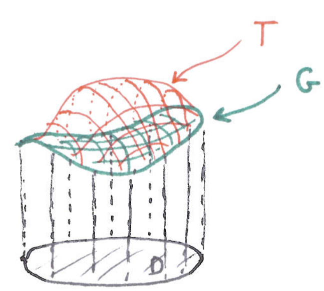
Notice that

$$\langle \vec{G}, \phi \rangle = 1 \quad \text{on} \quad G_f^D \qquad (**)$$

and that (an easy calculation shows that)

$$d\phi = 0$$

Now we consider some other 2-current $T \ni \partial T = \partial G$.



① $T - G$ encloses a region Ω,

② $(T-G)(\phi) = \int_\Omega d\phi = 0$

③ $G(\phi) = M(G)$ by $(**)$

④ $T(\phi) \leq M(T)$

Combining, we have $M(T) \geq M(G) \Rightarrow G$ is minimizing

... more on calibration next time.