PCMI Lecture #9

Pedagogical Lecture: $\int |\nabla u|^{p_1} + \lambda \int |u-f|^{p_2}$ and other toys

The "functional" or functional family

$$F_{p_1,p_2,\lambda_1,\lambda_2}(u) \equiv \lambda_1 \int |\nabla u|^{p_1} + \lambda_2 \int |u-f|^{p_2}$$

exhibits a wide variety" of ideas and technical facets, at least from the analysis and even PDE points of view.

The one key feature is the existence of a variety of algorithms, most rather simple to implement, for calculating minimizers. And, doing this in matlab, we have at our fingertips a laboratory in which students can explore and innovate.

Many concepts can be introduced and explored theoretically, while at the same time, computations using matlab, octave or python can be made to get a hands on feel for how these functionals behave in the function spaces they operate on... what minimizers are like, what the flows they generate look like, etc.

$F_{(p_1,p_2,\lambda_1,\lambda_2)}(u)$ $\quad$ $0 \le p_1, p_2 \le \infty$, $\quad 0 \le \lambda_1, \lambda_2 \le \infty$

E.L. equations ... easy to introduce to any undergraduate with calculus background... not often done

$$\boxed{F_{(p,0,1,0)}(u) = \int |\nabla u|^p \, dx} \overset{EL}{\Longrightarrow} p\nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right)$$

p-laplacian

$p=2 \to$ usual laplacian

$p=1 \to$ minimal surfaces

and $\mathbb{R}^2$ is interesting and simple at least in some ways

Key Point: all these are easily implementable in Matlab!

$\boxed{p = 1}$

$$\nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = 0 \quad \Rightarrow$$ level sets have zero mean curvature.

In $\mathbb{R}^2$ this can be played with by hand: how do you connect boundary values of equal value up with straight lines (or flat regions) in such a way as to minimize $\int |\nabla u|$?

While the analysis certainly reaches deeply, it is also very geometric and easy to grasp in statement and even in initial approach to solutions. This BV seminorm is a big part of the renaissance in mathematical image analysis. The paper that started it all (Rudin, Osher, and Fatemi) is accessible to undergraduates. $\frac{\partial u}{\partial t} = |\nabla u| \left(\nabla \cdot \frac{\nabla u}{|\nabla u|} \right)$ cool flow!

$\boxed{p = 2}$

E.L.

$$\nabla \cdot \nabla u = \Delta u \quad \rightarrow$$ heat kernel

$\boxed{\text{E.L. gradient flow}}$

$$\frac{\partial u}{\partial t} = \Delta u$$ most famous P.D.E.

can be used, is used ubiquitously... even for smooth images.

- Jones & Le scale space work is a new use.

$\boxed{p = \frac{1}{2}}$

has been used numerically, somewhat mysterious since this functional despises anything but jumps. Dimension of jump set is bounded by an expression in p.

$$\boxed{F_{(0,P,0,1)}(u) = \int |u-f|^p = \|u-f\|_p^p}$$

of course this norm is central to lots of analysis. And, for $p<1$ this turns out to have very interesting connections in compressed sensing, another theme that is currently hot and here at PCMI.

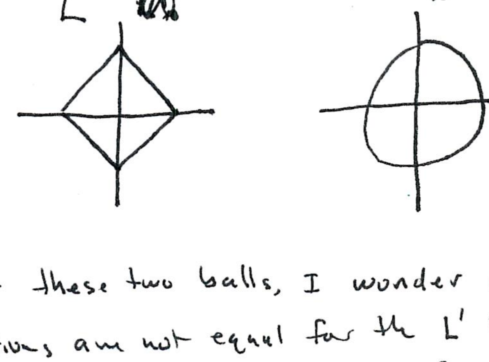
Of course minimizing this is rather boring... we just get $u=f$. But there are questions that lie close by and are not boring.

For example:

* $\min_u \|u-f\|_p^p$ for a wild f and $u \in$ {functions with bounded second derivative ... i.e. $u'' < C$, C chosen before hand}

* Loosening this up leads to discussion of function spaces dense in other function spaces ... like the density of smooth functions in the L^p spaces.

* Example exploration: comparing the L^1 and L^2 balls in $\mathbb{R}^n$ is our starting point.



looking at these two balls, I wonder about the fact that all directions are not equal for the L^1 ball i.e. its anisotropy. That is, using our L^2 distance on the L^1 ball (or vice versa) we find the coordinate axis are special in L^1...

Can we find an interpretation of this in function spaces?

Considering $L^1([0,1])$ and $L^2([0,1])$

(0) $L^1([0,1]) = \{u \mid \int_0^1 |u|\,dx < \infty\}$, $L^2([0,1]) = \{u \mid \int_0^1 |u|^2 dx < \infty\}$

(1) $L^2([0,1]) \subset L^1([0,1])$

(2) Define $\bar{u} = \int_0^1 |u|\,dx$

Compute:
$$\int_0^1 (|u| - \bar{u})^2 dx = \int_0^1 |u|^2 - 2\int |u|\bar{u}\,dx + \int \bar{u}^2 dx$$
$$= \int_0^1 |u|^2 - \int_0^1 \bar{u}^2 dx$$

$$\geq 0$$

$$\Rightarrow \quad \int_0^1 |u|^2 \geq \int_0^1 \bar{u}^2 dx$$

and therefore $\|u\|_2 \geq \bar{u} = \|u\|_1$

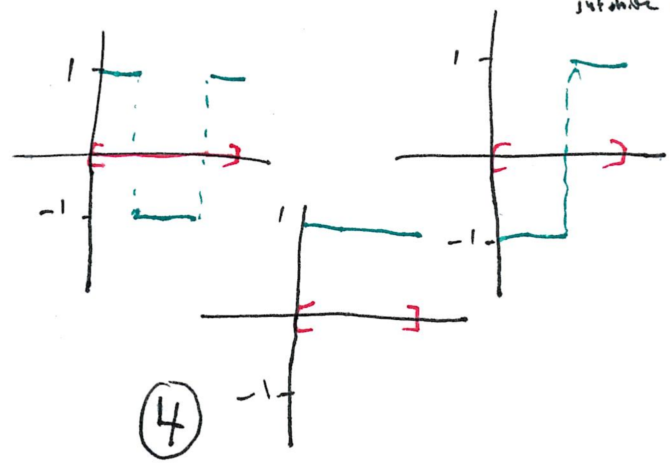
we also get from this calculation that

$\|u\|_2 = \|u\|_1$ only when $|u| = \bar{u}$

So on the L^2 unit ball ~~all~~ only functions $u : [0,1] \to \{-1, 1\}$ also have unit norm in L^1.

And that makes sense:

all are something that makes sense as infinite dimensional coordinate directions



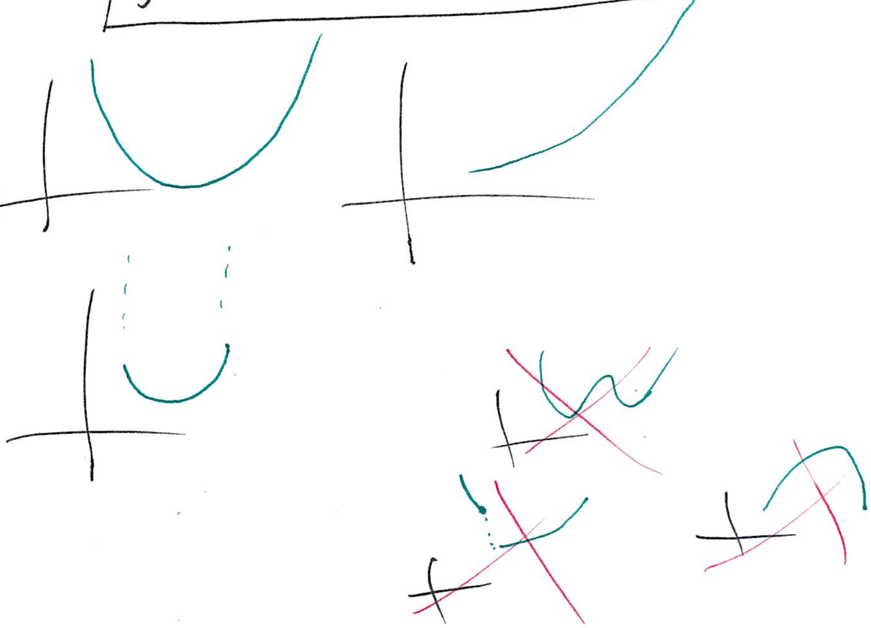
$$\boxed{F_{(2,2,1,\lambda)}(u) = \int |\nabla u|^2 + \lambda \int |u-f|^2}$$

This functional is the starting point for all other image functionals: and it is particularly nice.

One can illustrate all sorts of analysis ideas with this functional ad computations are easy, simple ... something that helps illuminate the nature of the functional.

$$\boxed{f(\alpha x + (1-\alpha) y) \le \alpha f(x) + (1-\alpha) f(y)}$$

convexity



show $\int |\nabla u|^2$ is convex:

> comment – it is never too early to get students thinking about inequalities. This is an example.

If we can show $|\alpha v + (1-\alpha) u|^2 \le \alpha |v|^2 + (1-\alpha) |u|^2$ we are done because we can integrate this pointwise inequality to get the one we want.

$$|\alpha v + (1-\alpha)u|^2 = (\alpha v + (1-\alpha)u)\cdot(\alpha v + (1-\alpha)u)$$

$$= \alpha^2 v\cdot v + 2(\alpha)(1-\alpha)u\cdot v + (1-\alpha)^2 u\cdot u$$

now subtract this from $\rightarrow \alpha\, v\cdot v + (1-\alpha)\, u\cdot u$ to get

$$(\alpha - \alpha^2)v\cdot v - 2(\alpha)(1-\alpha)\,u\cdot v + ((1-\alpha)-(1-\alpha)^2)\,u\cdot u$$

$$= (\alpha)(1-\alpha)\,(v\cdot v - 2u\cdot v + u\cdot u)$$

$$= (\alpha)(1-\alpha)\,(v-u)\cdot(v-u) \geq 0$$

$$\Rightarrow \quad |\alpha v + (1-\alpha)u|^2 \leq \alpha|v|^2 + (1-\alpha)|u|^2$$

$$\Rightarrow \quad \int |\alpha\nabla\phi + (1-\alpha)\nabla\psi|^2 \leq \alpha\int|\nabla\phi|^2 + (1-\alpha)\int|\nabla\psi|^2$$

$$\Rightarrow \quad \int|\nabla u|^2\,dx \text{ is convex}$$

In fact our proof above shows $\int|\nabla u|^2\,dx$ is strictly convex. And essentially the same proof shows that $\int|u-f|^2\,dx$ is also strictly convex.

Euler Lagrange

As already seen many times in this course, a natural step is to calculate the Euler Lagrange equation

~~[illegible]~~

In this case we get, in a way easy for students to follow... ~~[illegible]~~

$$\frac{d}{dt} F(u+tv) = \frac{d}{dt}\left[\int \nabla u\cdot\nabla u + 2t\,\nabla u\cdot\nabla v + t^2\,\nabla v\cdot\nabla v + \lambda\left(u^2 + 2uvt + t^2v^2 - 2(u+tv)f + f^2\right) \right]\Bigg|_{t=0}$$

$$= \int 2\,\nabla u\cdot\nabla v + 2\lambda uv - 2\lambda v f$$

$$= 2\int \nabla u\cdot\nabla v + \lambda(u-f)v$$

$\Downarrow$ integrate by parts

$v = 0$ on boundaries

$$= 2\int (\nabla\cdot\nabla u + \lambda(u-f))\, v\, dx$$

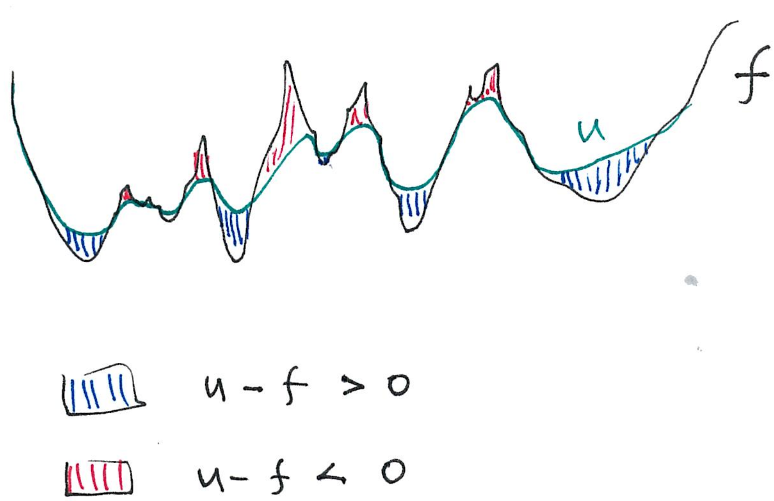
$\Rightarrow$ at a minimum a necessary condition is that

$$\boxed{-\Delta u + \lambda(u-f) = 0}$$

or

$$\Delta u = \lambda(u-f)$$

comment: geometry is almost always a good way to get students engaged with guiding intuition!



There a whole host of questions, all of which can be explored at an advanced undergraduate research level.

Here is a partial list:

* Existence of solutions?
* uniqueness?
* regularity?
* stability to perturbation of f
* dependence on λ
* behavior of various numerical methods for finding solutions.

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Comment: the sky is the limit here. All the nicest properties are present – convexity, linearity, L^2 everything – and this makes various pieces quite gentle.

of course ...

$$F_{(1,2,1,\lambda)}(u) = \int |\nabla u| + \lambda \int |u-f|^2$$

$$F_{(1,1,1,\lambda)}(u) = \int |\nabla u| + \lambda \int |u-f|$$

... are a rich sourse of exploration and a jumping off point for lots of research.

In fact: $\underset{\text{dwell there}}{(2,2,1,\lambda)} \to \underset{\text{see a "review"}}{(1,2,1,\lambda) + (1,1,1,\lambda)}$

great preparation for independ. research project

... ideas will arise ... And don't avoid, ~~reinvention~~ reinvention!