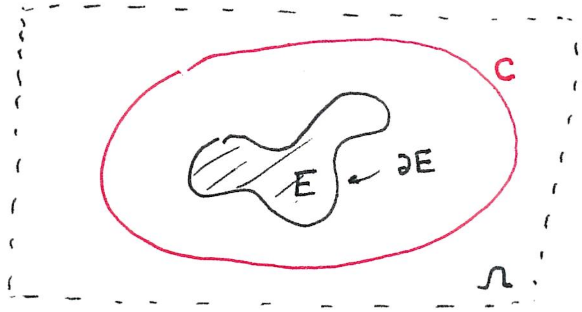
PCMI Lecture # 8

Curves Flows and Level set methods I

In this lecture we begin our exploration of curves and level sets in image al data analysis. We will Follow Chan & Vese 2001 quite closely.



$f = \chi_E$

Task: evolve C so that it coincides with ∂E

Snakes: Kass, Witkin and Terzopoulos 1988

$$C^* = \underset{C}{\operatorname{argmin}}\, F(C) = \alpha \int_0^1 |C'(s)|^2 ds + \beta \int_0^1 |C''(s)|\, ds - \lambda \int |\nabla f(C(s))|^2 ds$$

$\alpha, \beta, \lambda > 0$

First and Second terms: regularity of C

Third term: seeks hi gradient regions of f

First term: enforces uniform speed v

$$\int_0^1 v(t) = D \qquad \int_0^1 (v(t) - D)^2 dt \geq 0$$

$$\Rightarrow \int_0^1 v^2(t)\, dt \geq \int_0^1 D^2 dt$$

... so choose $v(t) = D$

Second term: this is just $|D^2 K|$ so this term encourages low curvature oscillation.

Third term: This encourages us to move the curve C to coincide with ∂E. (This term would, strictly speaking, force the curve to follow any discontinuity but in practice we do not have discontinuities since we would smooth the image a bit.)

Active Contour Method

$\boxed{\text{Ingredient 1}}$ **Edge Detectors:** Functions which go to 0 as the scalar argument goes to infinity which we apply to the norm of the gradient.

Example:

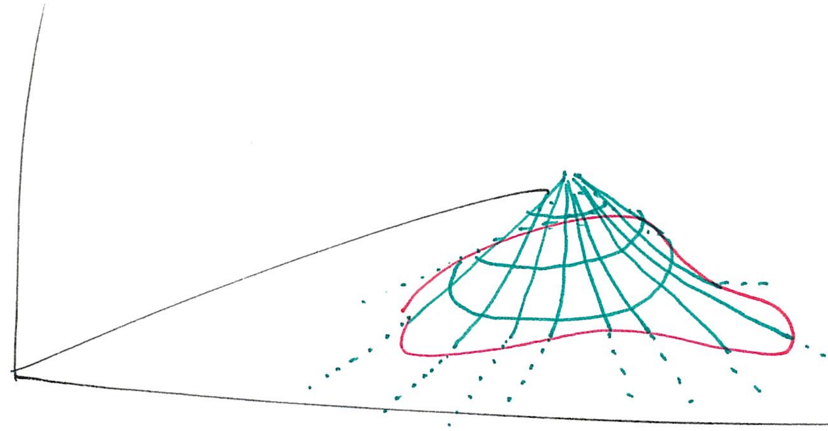
$$g(|\nabla f|) = \frac{1}{1 + |\nabla K_\sigma * f|^p} \qquad p \geq 1$$

note: we have moved the gradient operator to the smoothing kernel K_σ, σ is the width of the smoothing kernel.

(g is of course going to be zero at edges since $\nabla f = \infty$ there.)

$\boxed{\text{Ingredient 2}}$ **Level set method:**

Idea: represent the curve C implicitly, as the zero level set of a function ϕ.



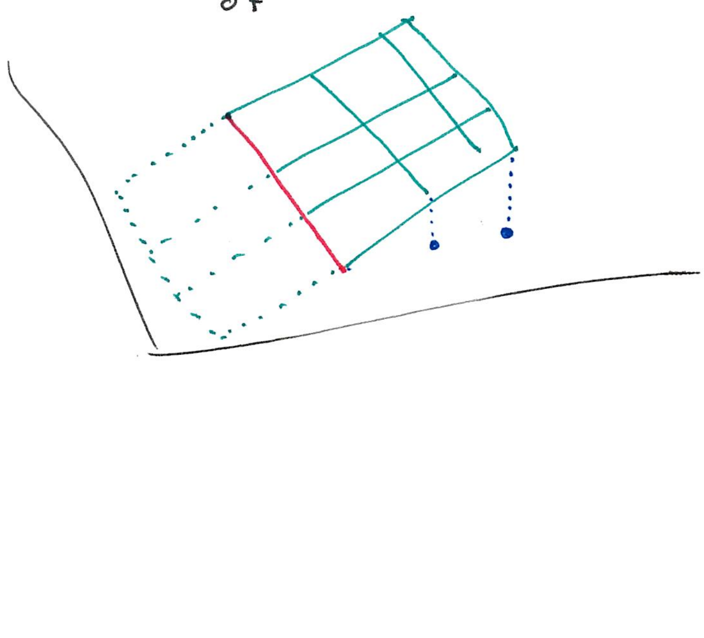
$C = \{x \mid \phi(x) = 0\}$

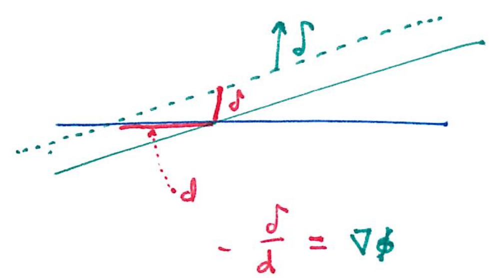
Suppose we want to move C with a vector field $\vec{V}(x)$. We can do that very simply and implicitly.

$$\frac{\partial \phi}{\partial t} = -\nabla\phi \cdot \vec{V} \quad \text{or} \quad \frac{\partial \phi}{\partial t} + \nabla\phi \cdot \vec{V} = 0$$

note that if $\vec{V}$ is a normal vector field then we can write

$$\frac{\partial \phi}{\partial t} + |\nabla\phi||\vec{V}| = 0$$





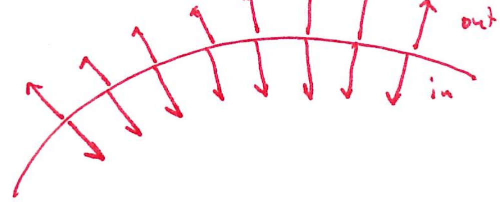
$$-\frac{\delta}{d} = \nabla\phi$$

$$\Rightarrow \quad \boxed{\dot{\delta} + \dot{d}\,\nabla\phi = 0}$$

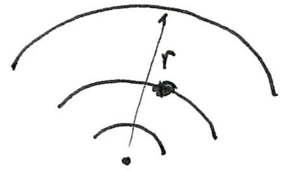
i.e. $\nabla\phi$ is the conversion factor needed to convert vertical motion to horizontal motion.

Ingredient 3

mean curvature



$$H = \nabla \cdot \frac{\nabla \phi}{|\nabla \phi|}$$



$\phi = \sqrt{x^2+y^2}$ = distance to origin

$$H = \nabla \cdot \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$$

$$= \frac{x^2+y^2 - x^2}{(x^2+y^2)^{3/2}} + \frac{x^2+y^2-y^2}{(x^2+y^2)^{3/2}}$$

$$= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

Likewise in $\mathbb{R}^n$ we get

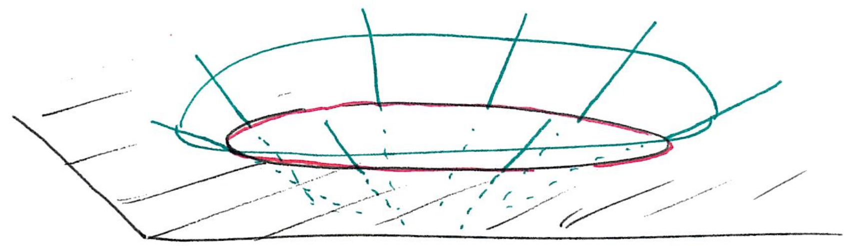
$$H = \nabla \cdot \left(\frac{r}{|r|} \right) = \frac{n-1}{r}$$

For the level set r from the origin.

Note: Some use $\phi < 0$ inside C others use $\phi > 0$ inside C. $\Rightarrow \frac{\nabla \phi}{|\nabla \phi|}$ pointing out, $\frac{\nabla \phi}{|\nabla \phi|}$ pointing in, respectively

Choosing $\phi > 0$ outside:

$$\frac{\partial \phi}{\partial t} = g(|\nabla f|)\,|\nabla\phi|\left(\nabla\cdot\frac{\nabla\phi}{|\nabla\phi|} + \nu\right)$$



in this case $\frac{\nabla\phi}{|\nabla\phi|}$ is pointing out and $\nabla\cdot\frac{\nabla\phi}{|\nabla\phi|}$ is positive for this curve C. This makes this particular curve shrink and will continue to shrink even if there is a bit of negative curvature as long as $\nu > 0$, until $g(|\nabla f|)$ stops it.

It is (I believe) a little more typical to choose $\phi > 0$ inside.

One other model: geodesic model of Caselles, Kimmel, and Sapiro

$$F(c) = \int_0^1 |c'(s)|\cdot g(|\nabla f(c(s))|)\,ds$$

$$\Downarrow$$

level set formulation

$$\Downarrow$$

$$F(c) = \int g(|\nabla f|)|\nabla\phi|\,dx$$

$$\Downarrow$$

EL ...

$$\nabla\cdot\left(g(|\nabla f|)\frac{\nabla\phi}{|\nabla\phi|}\right)$$

To get propagation speeds invariant to ϕ, we move according to

$$\frac{\partial \phi}{\partial t} = |\nabla \phi| \left(\nabla \cdot \left(g(|\nabla f|) \frac{\nabla \phi}{|\nabla \phi|} \right) + V \cdot g(|\nabla f|) \right)$$

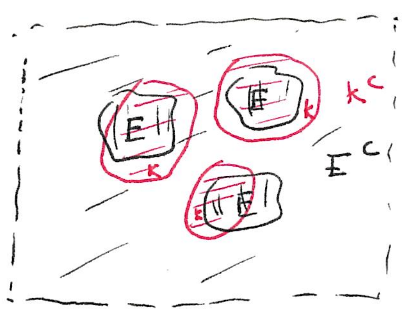
Chan-Vese : Active contours without edges

Inspiration here is the mumford shah model

recall

$$F(u, \Gamma) = \alpha \int_{\Omega} |u - f|^2 + \beta \int_{\Omega \setminus \Gamma} |\nabla u|^2 + \gamma \mathcal{H}^{n-1}(\Gamma)$$

And they consider piecewise constant images...



$$f = \begin{cases} c_1^* & \text{on } E \\ c_2^* & \text{on } E^c \end{cases}$$

$$F_0(K, c_1, c_2) = \int_K (f - c_1)^2 \, dx + \int_{K^c} (f - c_2)^2$$

$K = E$, $c_1 = c_1^*$, $c_2 = c_2^*$ is the unique minimizing set.

The complete Chan-Vese model is:

$$F_{cv}(K, c_1, c_2) = \alpha \mathcal{H}^1(C) + \beta\,(\text{area inside } C) + \gamma_1 \int_{\text{inside}(C)} |f - c_1|^2 \, dx + \gamma_2 \int_{\text{outside}(C)} |f - c_2|^2 \, dx$$

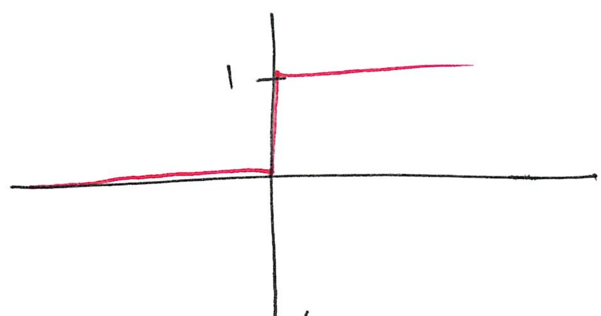
The Mumford Shah model, specialized to the case of piecewise constant outputs, is

$$\alpha \mathcal{H}^1(C) + \gamma_1 \int_{\text{in}(C)} |f - \bar{f}_{ic}|^2 + \gamma_2 \int_{\text{out}(C)} |f - \bar{f}_{oc}|^2$$

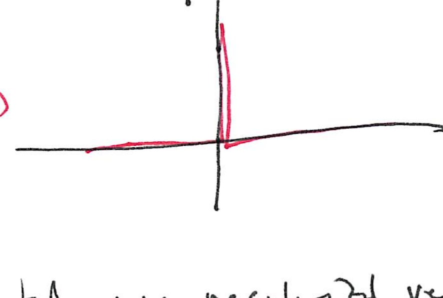
which is just F_{cv} with $\beta = 0$.

Level set formulation of the Chan-Vese model

Heavyside Function $H(x)$



$\frac{d}{dx} H(x) = \delta(x)$ (a measure)



(We will, upon implementation, use regularized versions of H & δ, H_ε, δ_ε $\quad \delta_\varepsilon = \frac{d}{dx} H_\varepsilon$)

Now we represent C by $\phi = 0$. $\phi > 0$ on the inside is the convention we use.

We note that

$$\mathcal{H}^1(C) = \int |\nabla \chi_K|\, dx$$

and $H(\phi) = \chi_K$, so we get

$$\mathcal{H}^1(C) = \int |\nabla H(\phi)|$$

$$= \int \delta(\phi)|\nabla\phi|$$

This makes sense by considering the regularized version, or a change of variables for the δ distribution.

$$\text{area in } C = \int H(\phi)\, dx$$

Regularized version!

for smooth δ_ε, H_ε this is the chain rule.

and

$$\int (f-c_1)^2 H(\phi)$$

$$\int (f-c_2)^2(1-H(\phi))$$

are the last two terms we need.

change of variables

$$\int \delta(Kx) f(x)\, dx$$

$y = Kx$

$dy = K\,dx$

$$\int \delta(y) f\left(\tfrac{y}{K}\right)\frac{dy}{K}$$

$$= \frac{f(0)}{K}$$

$\Rightarrow \delta(Kx)K = \delta(x)$

$$\Rightarrow F_{CV}(\phi, c_1, c_2) = \alpha\int \delta(\phi)|\nabla\phi| + \beta\int H(\phi)\,dx + \gamma_1\int (f-c_1)^2 H(\phi)\,dx + \gamma_2\int (f-c_2)^2(1-H(\phi))\,dx$$