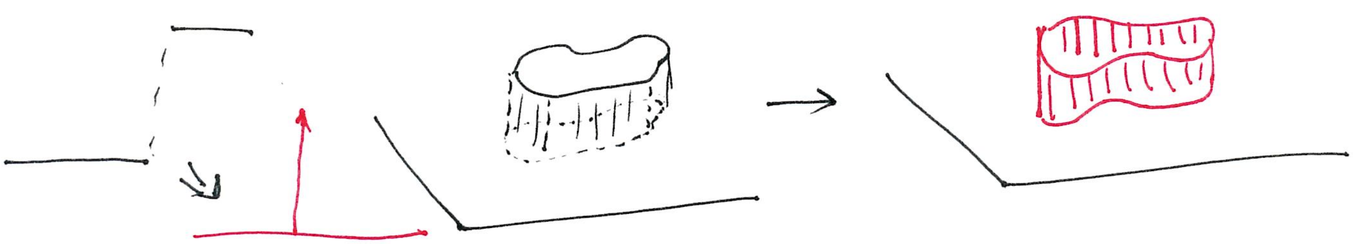
PCMI Lecture #15

Pedagogical Lecture: Boundaries, level sets, and distance functions: bare-handed experiments and computation.

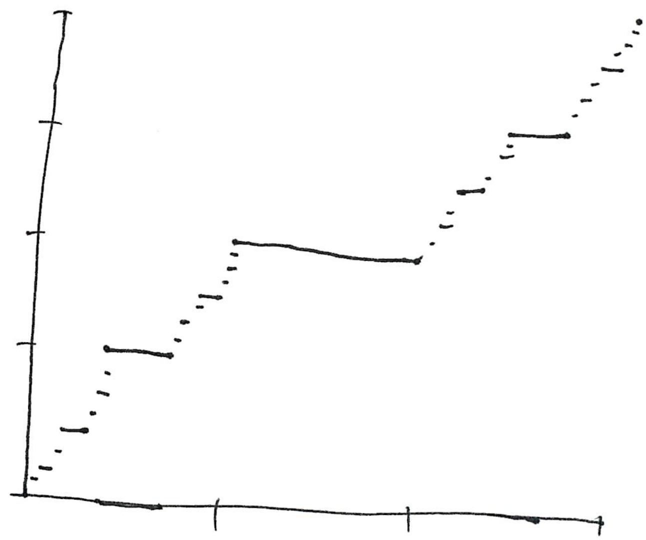
(C) We have seen that boundaries generate measures

i.e. $\nabla \chi_\Omega = \text{"}\delta(x-\partial\Omega)\text{"}$



Q: what else can we find by computing the gradient of a BV function?

A: Cantor function



f_c continuous

$\nabla f_c = 0$ a.e.

∇f_c sing w.r.t $\mathcal{L}^1$

$\int_0^1 |\nabla f_c| = 1$

⋮

f_c monotone

⋮

very useful example $\Longrightarrow$ Mumford Shah

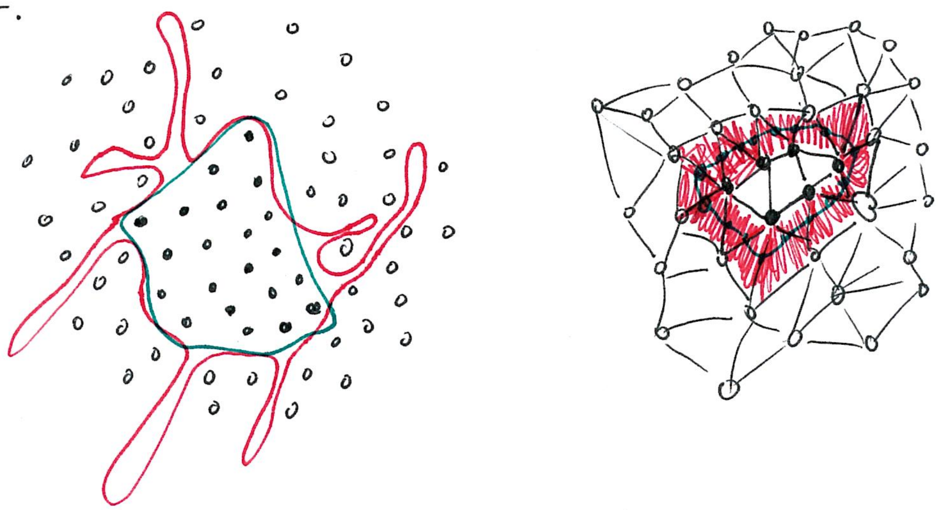
$\Downarrow$

Ambrosio invented SBV to deal with BV $-$ f_c

Pause:

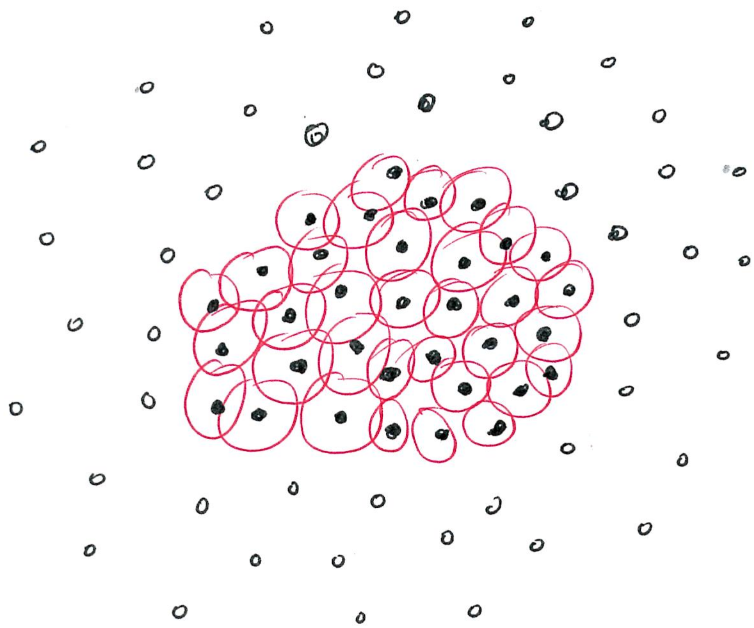
In this mode, one will encounter ideas which naturally suggest results that are advanced. The primary thing is to have an active, intuitive connection with the theorems, definitions, and conjectures, not necessarily to understand, at first pass, all the details of the entire proof of the theorem, or range of the implications of the conjecture or all possible examples of the definition. Active, intuitive connection <u>versus</u> obsessive comprehensive, exhaustive, detailed knowledge. Direction of motion in explorations will indicate which things demand complete exhaustive study and understanding versus what can be noted, reviewed intuitively at some (various) level of detail and stored away for a time when further study is necessary. Follow, leverage the Passion - "living path" mode

(D) Finding boundaries given points in and out of set.



This is a beautifully large place to explore, invent.

Q: what is the most regular boundary that satisfies the constraints of the data?



* at each • center a radial basis function, normalized (gaussian) add them all up and then threshold.

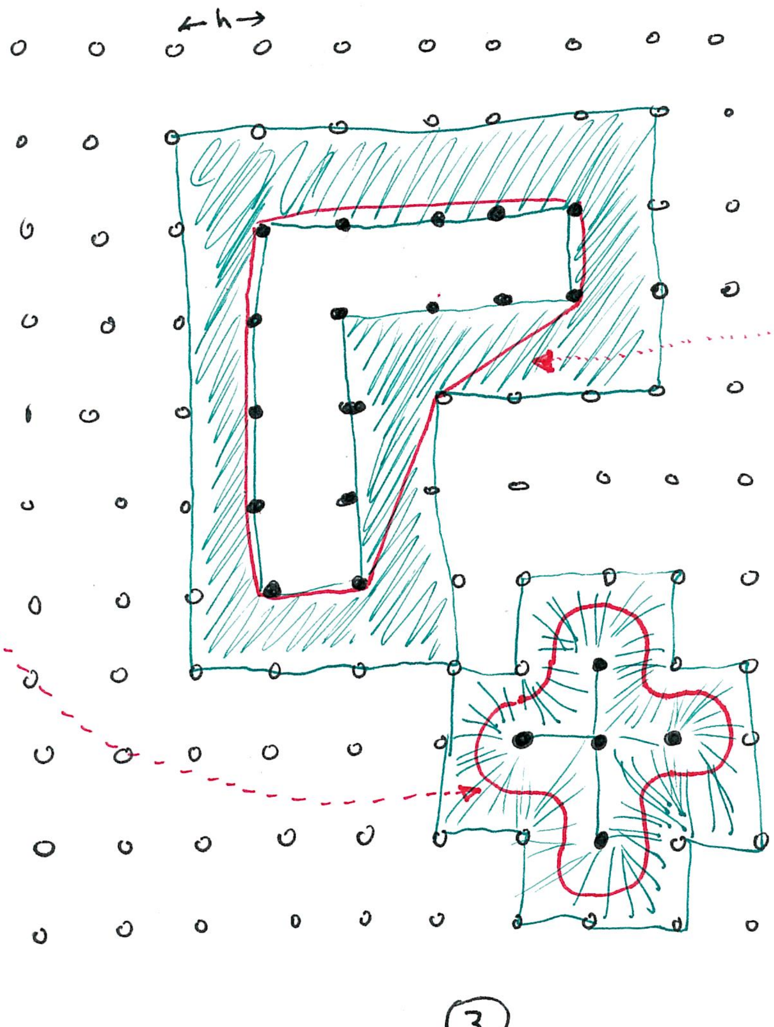
* at each • place a rbf and at each o place a negative rbf then threshold at 0.

* ... etc

We can start by studying the data on a regular grid problem

* can you find a $\gamma \ni \kappa \leq \frac{2}{h}$?

A: yes, thread the curve between inner and outer balls centered on grid points, balls with radius $= \frac{h}{2}$



* $\min_\gamma \int |\kappa|$?

* $\min_\gamma \oint |\kappa(s)|$?

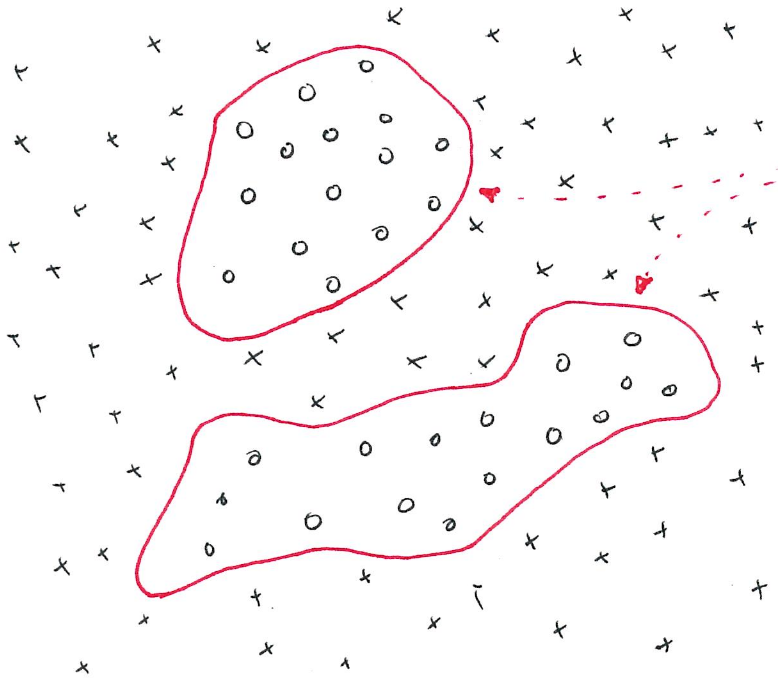
* $\min \int |\ddot{\gamma}| \, dt$?

A: there

* how regular are the level sets of the distance function?

A: if we consider unions of balls centered at inside points and choose a uniform radius r then we get the r level set. It is clear that this is only Lipschitz and for small enough r not even that.

This problem can be used to introduce Support Vector Machines and the associated Kernel trick or Kernel methods.



inverse image of the separating hyperplane in the Feature Space, the high dimensional space in which the separation becomes linear.

List of topic this rich problem motivates (unordered list)

- distance functions
- curvature
- arc length and constraint minimal arc length
- support vector machines and classification problems
- Kernel density estimation
- interpolation
- reach
- Topology of unions of balls centered on data
  ⟹ topological data analysis
- regularity of curves
- level set methods: choosing level sets of sums of kernels or of the distance function etc. is directly part of level set ideas which define curves implicitely thereby avoids topological problems.

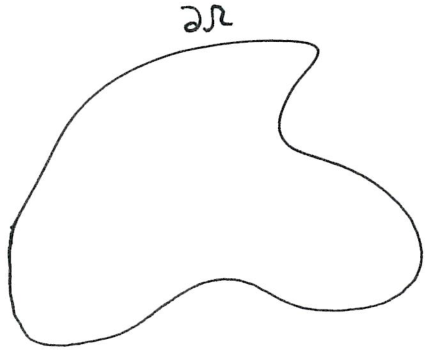
(E) **Problem:**

(a) Define neighborhoods of shapes

Background Problem: Propagation of uncertainty in shape & image processing pipelines.

(b) Define parameterizations of ~~[illegible]~~ some family of shapes containing some particular shape Ω

(c) Define metric on the parameterized family and use this to define ε-balls centered on Ω.



This problem is wide open, the only well studied system by "Random ~~shapes~~ sets" w "Random closed sets" used in Stochastic Geometry. But these are not what we want... we need parameterization

i) generate the distance function to $\partial\Omega$, perturb this

ii) generate normal perturbations $\Rightarrow$ reach important here

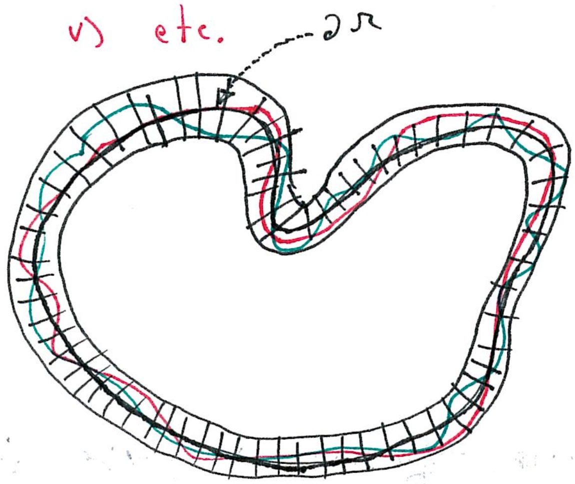
iii) generate a star-like radiating body to do perturbations with

iv) do i) but instead of choosing 0 level set always, look at the one-dimensional family of shapes obtained by ~~[illegible]~~ letting level y vary.

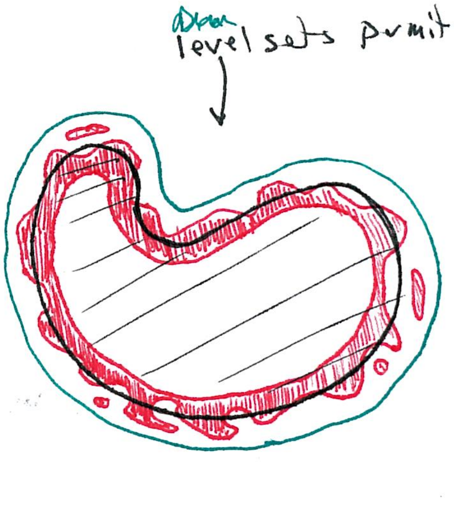
v) etc.

~~[illegible]~~

normal perturbations. →



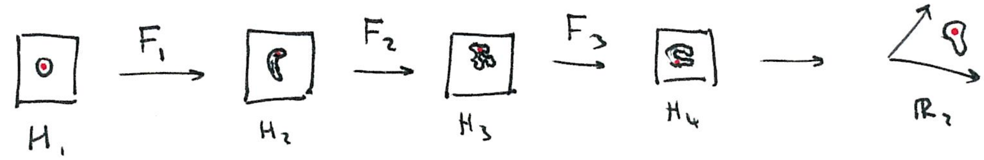
~~[illegible]~~ level sets permit this



To the list of things, this problem permits you to add

- stochastic processes in shapes
- Propagation of uncertainty in shape and image functionals.

This is a big deal

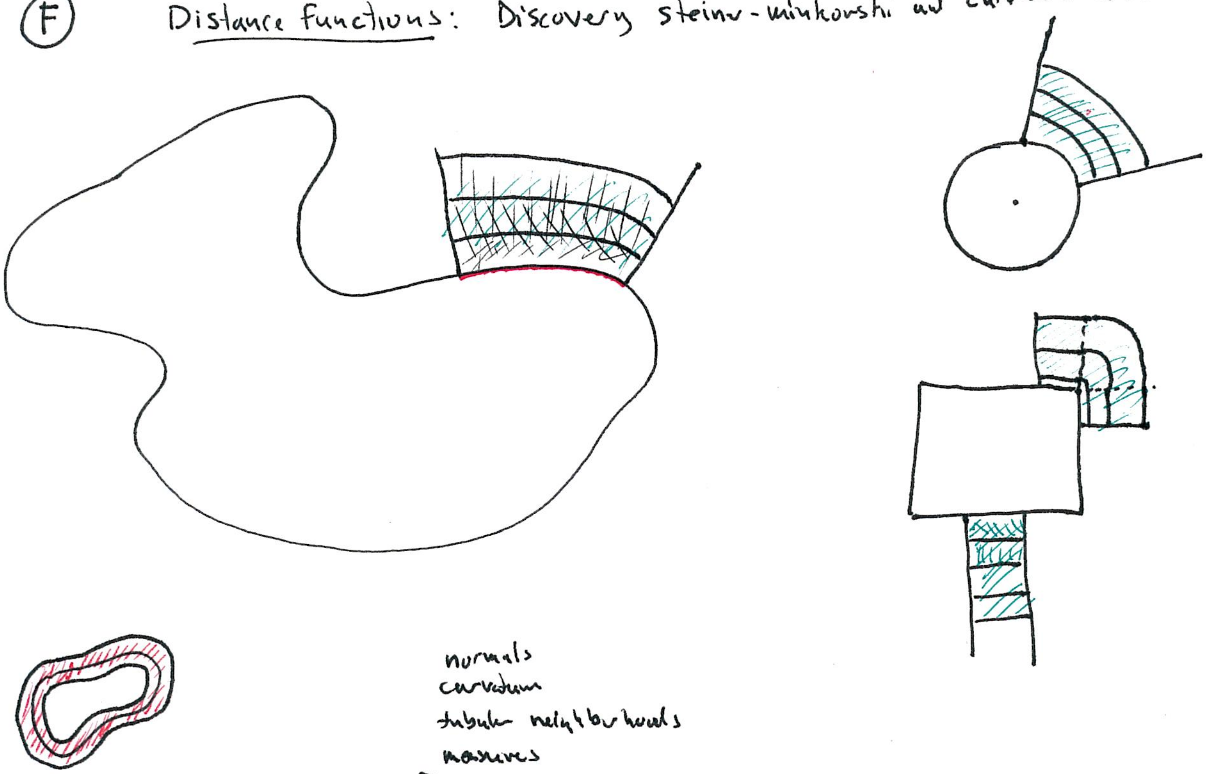


propagations

lots of room for exploration + innovation!

Calculations require reduced dimensional models of uncertainty...

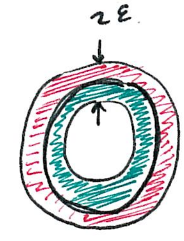
(F) Distance functions: Discovery steiner-minkowski and curvature measures



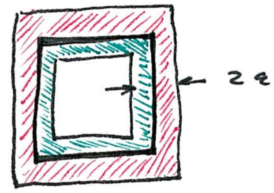
normals
curvature
tubular neighborhoods
measures
radon measures
regularity
reach
singular measures
steiner-minkowski
convex
distance functions
curvature measures

The First Steps...

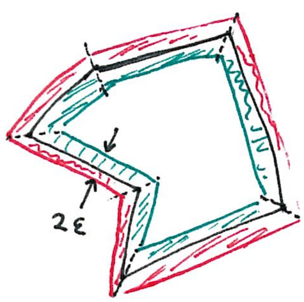
(a) Start by computing area in outer, inner, and symmetric tubes about circles and squares and triangles



= outer
= inner
+ = symmetric



(b) consider polygonal shapes other than squares



how does the turning angle relate to the area surplus or deficit in or out of the polygon?

how about the same question for partial circles... circular arcs

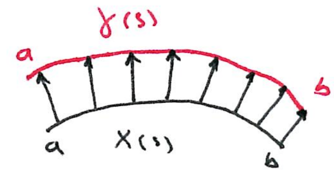
symmetric tube area as a function of ε?

(c) write a matlab program to calculate areas of ε tubes, neighborhoods of any input object. study the V(ε) plots.

very good exercise... some may prefer to work analytically

(d) since we generate inner and outer curves through the normal map

$$\gamma(s) = X(s) + \varepsilon n(s)$$
or
$$X(s) - \varepsilon n(s)$$



we can calculate

$$\mathcal{H}^1(\gamma_a^b) = \int_a^b \|\gamma(s)\| \, ds$$

$$= \int_a^b \|\dot{X}(s) + \varepsilon \dot{n}(s)\| = \int_a^b (1 + \varepsilon \kappa(s)) \|T(s)\| \, ds$$

$$= \int_a^b (1 + \varepsilon \kappa(s)) \, ds$$

* use this to get ~~[illegible]~~ a formulas for areas of tubes (partial / full curves) (Steiner-Minkowski)

* what can we say about the square (inner? outer?)
  $\Rightarrow$ singular parts

* nudge them towards

$$\mu_1(A) = \int_{X \cap A} 1 \, ds$$

$$\mu_2(A) = \int_{X \cap A} \kappa \, ds \quad \Rightarrow$$ interest for convex and non smooth shapes.