PCMI Lecture #14

Pedagogical Lecture: TV regularization and sets of finite perimeter as a path into geometric measure theory.

These last two lectures will look at how to integrate teaching and research centered around geometric analysis and shapes/images. First though, I will go into my philosophy in these ~~matters~~ matters, to give you all a full context for the next two lectures.

Philosophy and Perspective

The teaching of mathematics is usually <u>discipline</u> focused. We teach analysis, ring theory, algebraic geometry, harmonic analysis, combinatorics, ..., so on and so forth. Eventually some fraction of us will study a particular equation, some some chosen system of inequalities... fixed mathematical object, but not all, since some continue to study structure and develop tools.

PDE's is often taught with an equation focus

What is rarely done is the teaching of mathematics <u>through</u> a focus on a particular equation, a particular question.

While it is proper and sometimes necessary to focus on tools, disciplines, background, I am going to suggest that much more be done through a focus on specific objects, equations, functionals. In my opinion having a balance, a point-counterpoint relation, between theory and tool based exposition and particular object/equation explorations will bring a wholistic sense, an awareness of context and an automatic motivation to the ~~learning~~ learning.

In the next two days I will follow a somewhat meandering path through subjects we have already been exposed to in this course, illustrating that a problem/equation based approach can give a very natural motivation and context to theory that is thus encountered or inspired.

This kind of teaching takes on a character much more akin to research, learning what we need as we go... or inventing it if we have to, or are inspired to do so.

Naturally alligned with this style is the mode of active Learning that is the inspiration behind the Moore method. (No need to get distracted by the excesses of the Moore method). Simply understanding the original inspiration for R.L. Moores method – R.L. covering up proofs to see how much he could invent on his own, therefore kicking himself out of passive, into ~~[illegible]~~ an active frame of mind and a spirit – is enough to set us on the right path.

There are <u>many</u> ways to move the student into the active mode, each of which is right for some and not for others... our system has too long focused on one option: the exercises in the back of the chapter, or sometimes interspersed throughout the chapter. (of course I exagerate, but not too much)

But there are many other ways to trigger the active mode:

Examples: (nothing very new here)

students create exercises based on the reading / theory.

students come up with multiple examples illustrating theorems / Definitions (counter-examples too)

students encouraged to extend / generalize ideas in course.

students Find papers relating to subject at hand and read and explain them to the class.

students write code implementing ideas, theory.

students write expository papers explaining what they are learning

students give a very large latitude in the problems the solve ... say 20 out of 100 must be chosen.

etc, etc, etc ...

My claim is that the problem / research mode of teaching makes this diversity of activity natural, not artificially imposed.

The last piece of this active problem/research style of teaching is the lab environment.

Lab environment

- open everything, open access, open source, creative commons...
  - fast prototyping via scripting languages
    Matlab, Python, Octave, Sage, ...
  - Place: Blackboards, Notebooks, computers (laptops)
    wikipages (public ⊕ private), ejournals
  - Teams: 2-3
  - Jam sessions: Regular, inclusive not too long Formally, open ended
  - workshops: 2-3 Days immersive, problem focused
  - playfulness: a very big deal.

* problem / research style of teaching, point-counterpoint complement to ~~the~~ theory
* Quasi Moore method: principle = active mode (~~passive mode~~)
* Lab environment: Bare-handed work with problems, code, ideas.

With that, here we go

(A) ROF: the original paper to use total variation regularization in image analysis for the denoising problem.

After having students read the paper, I would guide them through the following exploration

$\int |\nabla u|$ ... what does this mean

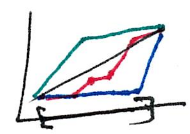
1D

i) C^1 u → not so hard

- calculate for some easy functions

~~[illegible]~~

ii) Lipschitz u → more interesting since we can now not get distracted by nonzero second derivatives



monotone with fixed B values have equal $\int |\nabla u|$

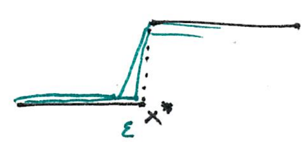
what are you tempted to conjecture? can you prove it?

Bring up the fact that it is often the case that an equation is used <u>before</u> it is well understood!!

what about the fact that ∇u not defined?
(case suggests these points don't matter <u>if</u> u is continuous)

connection: question how many points can a Lip function be discontinuous at?

iii) discontinuous u



ε → 0

⇐ this suggest
$\nabla u = \delta(x-x^*)$
⇒ what does this really mean?

iv) introduce weak definition

$$\int |\nabla u| \, dx = \sup_{|\phi| \le 1} \int \nabla u \cdot \phi$$

$$\int_\Omega \nabla u \cdot \phi = - \int_\Omega u (\nabla \cdot \phi) \qquad \text{compactly supported } \phi$$

$$\Rightarrow \int |\nabla u| = - \sup_{\substack{|\vec{\phi}| \le 1 \\ \vec{\phi} \ C^1}} \int u (\nabla \cdot \phi) \, dx$$

now define $\int |\nabla u|$ by the RHS when u not C^1. upshot it all makes sense even if u is discontinuous.

This is a general pattern seen over and over in analysis. To extend some idea, some calculation beyond where it should work, do something that is defined in that "outer region" that is equivalent for nice cases.

v) To understand

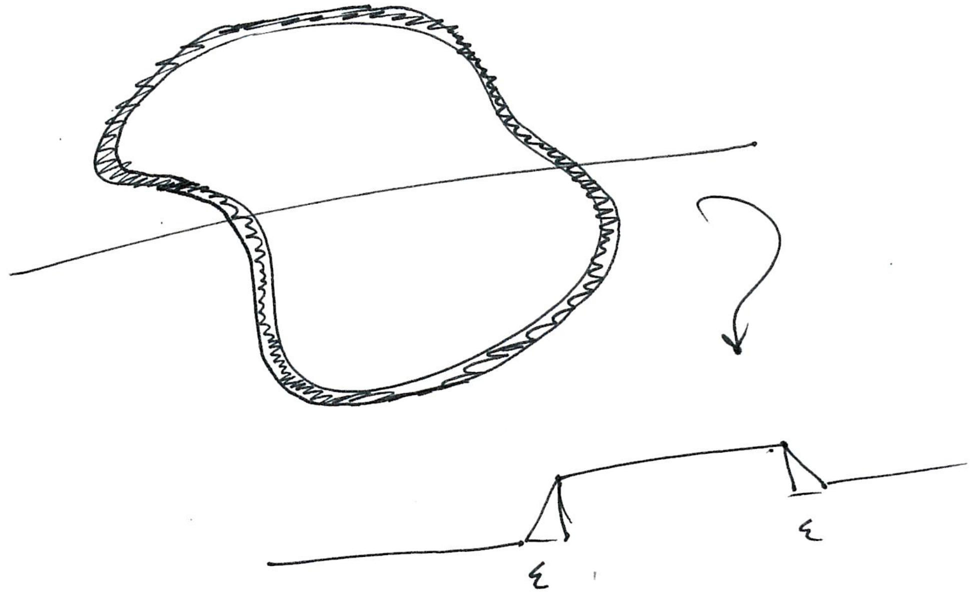
$$\nabla u = \vec{\sigma} \, d\mu$$

we need to understand

outer measure
measurable
regular
Borel
Borel regular
Radon

... lots of theory, very well motivated...
I am suggesting lots of student generated questions and answers along the way.

vi) more: 2D $\qquad \int |\nabla \chi_\Omega| = Per(\Omega)$



Q: how wiggly can $\partial\Omega$ be and still have finite length? $\Rightarrow$ intro to Rademacher's Theorem.

$\Rightarrow$ Reduced boundary, $\Rightarrow$ struct. thm. Caccioppoli sets.

Q: What other ways might we find $Per(\Omega)$?

what about $\dfrac{Vol(\Omega+\epsilon B) - Vol(\Omega)}{\epsilon}$?

vii) BV coarea in 2D $\qquad \int |\nabla u| = \int \mathcal{H}^1(u=y)\,dy$

picture!!

$\Downarrow$

usual coarea formula (works for Lipschitz)

$\Downarrow$

Fubini: special case of coarea

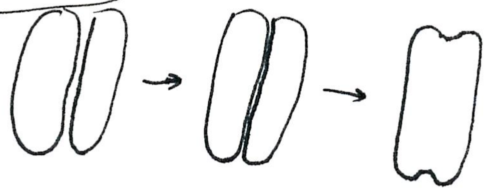
... a transition now

B Q: Can you find a sequence of shapes $\Omega_i \xrightarrow{L^1} \Omega$ such that $Per(\Omega_i) \to L > Per(\Omega)$?

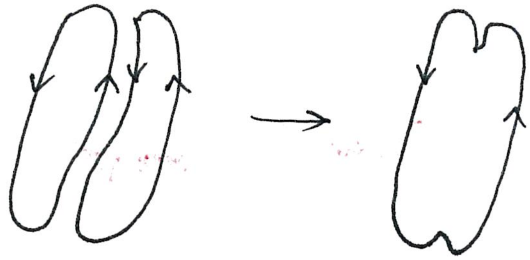
Q: how about " " " $L < Per(\Omega)$?

$\Longrightarrow$ lower semicontinuity for BV.

$\boxed{\int f \,\mathrm{div}\, g = \lim_{i\to\infty} \int f_i \,\mathrm{div}\, g \le \liminf_{i\to\infty} \int |Df_i|}$

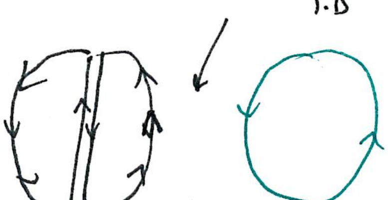


Boundaries make this natural



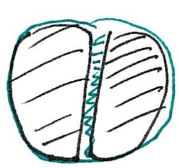
... somehow it seems natural that close, oppositely oriented boundaries should cancel.

Q: how can we do this ... construct a distance that says these shapes are close?
1-D ↑ that are boundaries

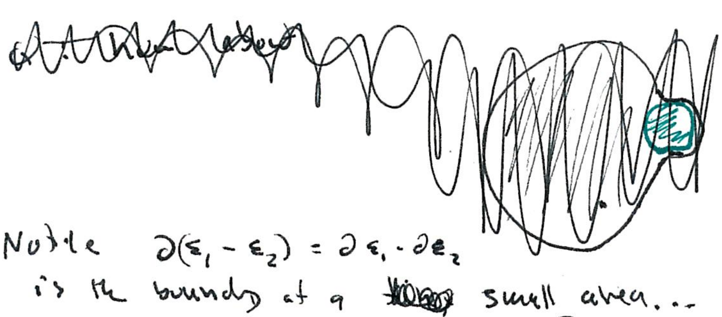


$\gamma_1 = \partial\Sigma_1$ $\gamma_2 = \partial\Sigma_2$

Q: how about comparing the shapes that these are boundary of?



distance = area of $|\Sigma_1 \Delta \Sigma_2|$



Note $\partial(\Sigma_1 - \Sigma_2) = \partial\Sigma_1 - \partial\Sigma_2$
is the boundary of a ~~[illegible]~~ small area... key observation.

Inspiration: measure size of $\gamma_1 - \gamma_2 = \partial(\Sigma_1 - \Sigma_2)$ by area of $\Sigma_1 - \Sigma_2$. that is, for boundaries T (in our case $T = \partial(\Sigma_1 - \Sigma_2)$) where $T = \partial S$

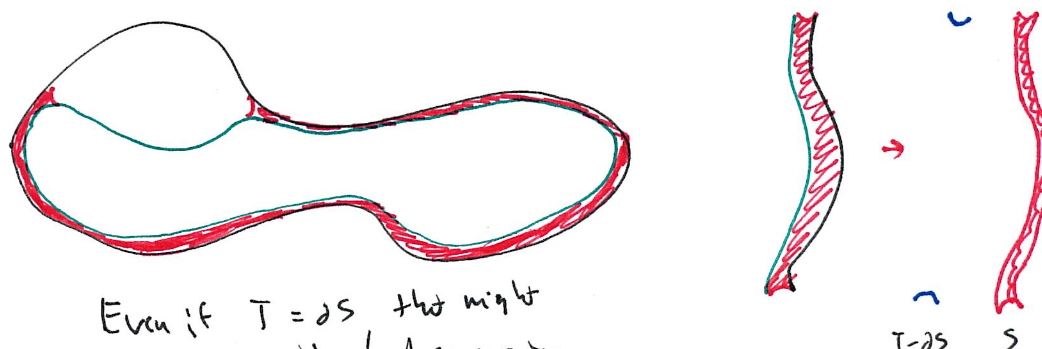
$$\mathbb{F}(T) = \mathcal{H}^2(S)$$

Q: what if we want to apply this to $T \neq \partial S$ for any S? well, try

$$\mathbb{F}(T) = \underbrace{\mathcal{H}^1(T - \partial S)}_{\text{length of miss}} + \underbrace{\mathcal{H}^2(S)}_{\text{area of approx diff.}}$$

... minimize over S.

→ Bingo: Flat norm... generalized to any current not-just boundaries.



Even if $T = \partial S$ this might not be optimal decomposition

... moving on to more boundaries next time.