# PCMI Lecture #13

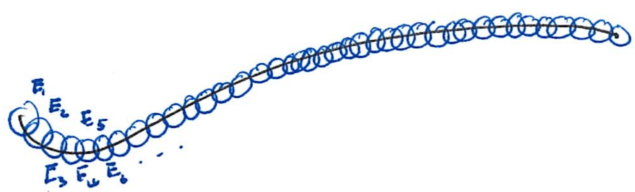
Covers and Neighborhoods: Hausdorff measures and Steiner-Minkowski

In this lecture I will look more carefully at Hausdorff measure, a measure that ~~is~~ ubiquitous in careful treatments of the geometric analysis connected to data and image analysis, and the Steiner-Minkowski formula, for its intrinsic beauty and its usefulness for shapes.

## Hausdorff Measure

$$\mathcal{H}^K_\delta \equiv \left\{ \inf \left\{ \sum_{E_i \in \mathcal{E}} \alpha(K) \left( \frac{\operatorname{diam}(E_i)}{2} \right)^K \;\middle|\; \sup_{E_i \in \mathcal{E}} \operatorname{diam}(E_i) \leq \delta \right\} \right.$$

$$\mathcal{H}^K \equiv \lim_{\delta \to 0} \mathcal{H}^K_\delta = \sup_{\delta > 0} \mathcal{H}^K_\delta$$




* $\mathcal{H}^K$ is a Borel regular measure
* $\mathcal{H}^n = \mathcal{L}^n$ in $\mathbb{R}^n$

## Hausdorff Dimension

$$d_H = \dim(E) \equiv \inf\{0 \leq s < \infty \mid \mathcal{H}^s(E) = 0\}$$

$$0 \leq \mathcal{H}^{d_H(E)}(E) \leq \infty$$

Examples can take any value in that range.

Hausdorff measure generalizes $\mathcal{L}^n$ to manifolds (tame) and rectifiable (wilder) sets and fractals (wild!). And a key point is that $k$ need not be an integer.

**more on dimension**

## Densities

$$\rho_k^*(\mu) \equiv \limsup_{r \to 0} \frac{\mu(B_r(x))}{\alpha(k)\, r^k}$$

$$\rho_{*,k}(\mu) \equiv \liminf_{r \to 0} \frac{\mu(B_r(x))}{\alpha(k)\, r^k}$$

$$\rho_k(\mu) \equiv \rho_k^*(\mu) = \rho_{k,*}(\mu) = \lim_{r \to 0} \frac{\mu(B_r(x))}{\alpha(k)\, r^k}$$

when this holds

(1)

$\mu = \mathcal{L}^2 \llcorner E$


$\rho_2(\mu)(x_1) = 0$

$\rho_2(\mu)(x_2) = \frac{1}{2}$

$\rho_2(\mu)(x_3) = \frac{1}{4}$

(2)

$$E \equiv \bigcup_{i=1}^{\infty} A_i \qquad \mu = \mathcal{L}^2 \llcorner E$$

$$A_i = B_{r_i} \setminus B_{r_i/2^i}$$

$$r_{i+1} \leq \frac{r_i}{2^{2i}}$$

Then $\rho_2^*(\mu)(0) = 1$

$\rho_{2,*}(\mu)(0) = 0$

(3) if $\mathcal{H}^k(E) < \infty$, $E$ is $\mathcal{H}^k$ measurable, $E \subset \mathbb{R}^n$

then

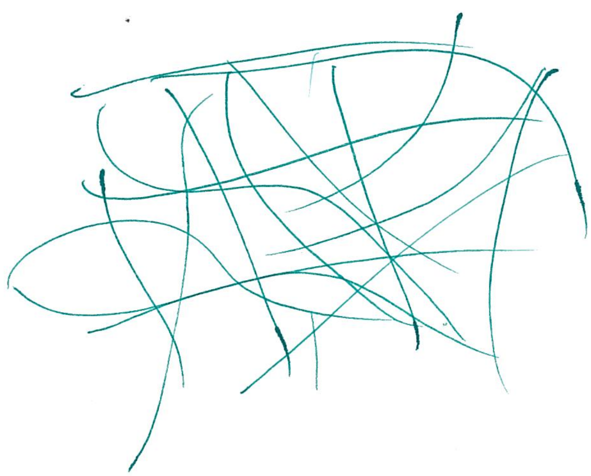
$$\frac{1}{2^k} \leq \limsup_{r \to 0} \frac{\mathcal{H}^k(B_r(x) \cap E)}{\alpha(k)\, r^k} \leq 1$$

## rectifiable sets

$E \subset \mathbb{R}^n$ is k-rectifiable if there is a countable family of ~~compact~~ lipschitz maps $f_i : \mathbb{R}^k \to \mathbb{R}^n$ $\ni$ $\mathcal{H}^k(E \setminus \bigcup_i f_i(\mathbb{R}^k)) = 0$.

E is k-rectifiable if it is a subset, up to a measure 0 ($\mathcal{H}^k$) error, of a countable family of k ~~[illegible]~~ dimensional lipschitz submanifolds of $\mathbb{R}^n$.

Comment: We can take these submanifolds to be $C^1$ ... this is a theorem.



"nice" rectifiable set

Theorem: $\rho_k(\mathcal{H}^k \llcorner E) = 1$ for $\mathcal{H}^k$ almost all of E when E is k-rectifiable and $\mathcal{H}^k(E) < \infty$

(comment all we need is local finiteness)

# Preiss' Big Result

Marstrand

Let $\mu$ be a locally finite measure on $\mathbb{R}^n$ and $k$ a non-negative real number.

assume $\lim_{r \downarrow 0} \frac{\mu(B_r(x))}{r^k}$ exists, is finite and is nonzero $\mu$ a.e. $x$

Then: either $\mu = 0$ or $k$ is a non-negative integer $\leq n$.

Preiss

In the latter case, $\mu$ satisfies this requirement

iff $\exists$ Borel measurable $f$ and $\{\Gamma_i\}$, a countable collection of Lip $k$-dim' submanifolds of $\mathbb{R}^n$ $\ni$

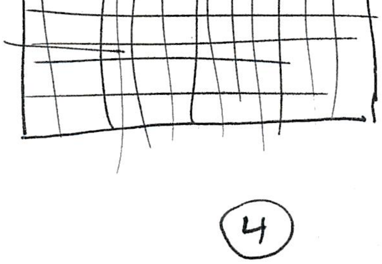
$$\mu(A) = \sum_i \int_{\Gamma_i \cap A} f(x)\, d\mathcal{H}^k(x)$$

for any Borel $A \subset \mathbb{R}^n$.

Consequence: Fractals do not have well behaved densities.

Also ... $\rho_k(\mu) = f$ $\mu$ a.e. $x$.

Example:



$f_i = \frac{1}{2^i}$ $\{\Gamma_i\}$ = vertical or horizontal lines with rational intersection with $x$ or $y$ axis.

Zooming in, we eventually only see the line segments we are on!

## Hausdorff Summary

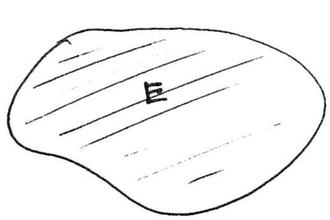
For careful treatment of a large range of data and image analysis methods and models, Hausdorff measure is the right tool.

There is a great deal that can be understood by thinking of $\mathcal{H}^k$ as $\mathcal{L}^k$ that "bends around", but there is also lots of fine detail to be careful with and explore.

Rectifiable sets are natural models for wild phenomena (non-fractal) yet with enough structure to enable analysis.

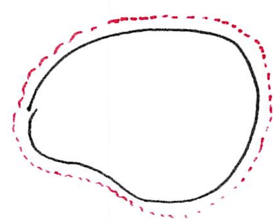
## Steiner Minkowski

Minkowski Sum in a Vector Space




$+ \quad \varepsilon B_r(0)$


$E + \varepsilon B_r$ :



= all points a distance $\varepsilon$ or less from E.

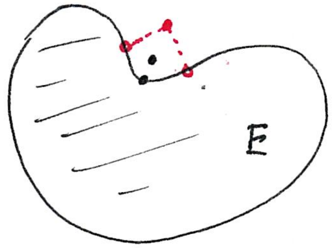
<u>Question:</u> What is $\mathcal{H}^n(E + \varepsilon B_r)$ ?

Surprising Answer: an $n$th order polynomial in $\varepsilon$ whose coefficients depend on $E$.

assumptions: either $E$ is convex or reach$(E) < \varepsilon$

Reach

reach is "farthest" you can be from the set $E$ and have a unique closest point... actually the sup of distances for which the closest point is unique.




$\mathbb{R}^n$

Careful definition

$d(x) \equiv$ distance from $x$ to $E$.

$N(x) \equiv |\{y \in E \mid |x-y| = d(x)\}|$
$= \mathcal{H}^0(\{y \in E \mid |x-y| = d(x)\})$

reach $\varepsilon$ is biggest $\varepsilon \ni$ if $d(x) < \varepsilon \Rightarrow N(x) = 1$

If $\partial E$ has normals $\Rightarrow$ normals do not intersect until at least a distance $\varepsilon$ from $E$.

The formula: First for $\mathbb{R}^3$

$$\mathrm{Vol}(E + \varepsilon B) = \mathrm{Vol}(E) + \varepsilon\,\mathrm{Area}(\partial A) + \frac{\varepsilon^2}{2}\int_{\partial A} H \, d\mathcal{H}^2 + \frac{\varepsilon^3}{3}\int_{\partial A} G \, d\mathcal{H}^2$$

(actually in 2D, we don't have last term and 3rd term is simply $\frac{\varepsilon^2}{2} \cdot 2\pi$ when $E$ is topologically trivial)

In general:

$$\mathcal{H}^n(E+\varepsilon B) = \mathcal{H}^n(E) + \varepsilon \int_{\partial E} 1 \, d\mathcal{H}^{n-1}(x)$$

$$+ \frac{\varepsilon^2}{2} \int_{\partial E} \Sigma \kappa \, d\mathcal{H}^{n-1}(x)$$

$$\cdots + \frac{\varepsilon^{k+1}}{k+1} \int_{\partial E} \sum_{S \in S(k)} \prod_{i \in S} \kappa_i \, d\mathcal{H}^{n-1}(x)$$

$$\cdots + \frac{\varepsilon^n}{n} \int_{\partial E} \prod_{i=1}^{n-1} \kappa_i \, d\mathcal{H}^{n-1}(x)$$

why is this interesting for applications?

The coefficients of the polynomial are invariant shape signatures, intrinsic volumes, and are the integrals of the curvature measures.

Proof: Slides

go to slides