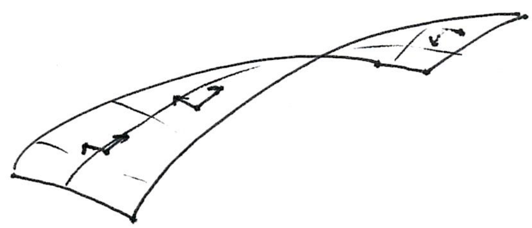
PCMI Lecture # 12

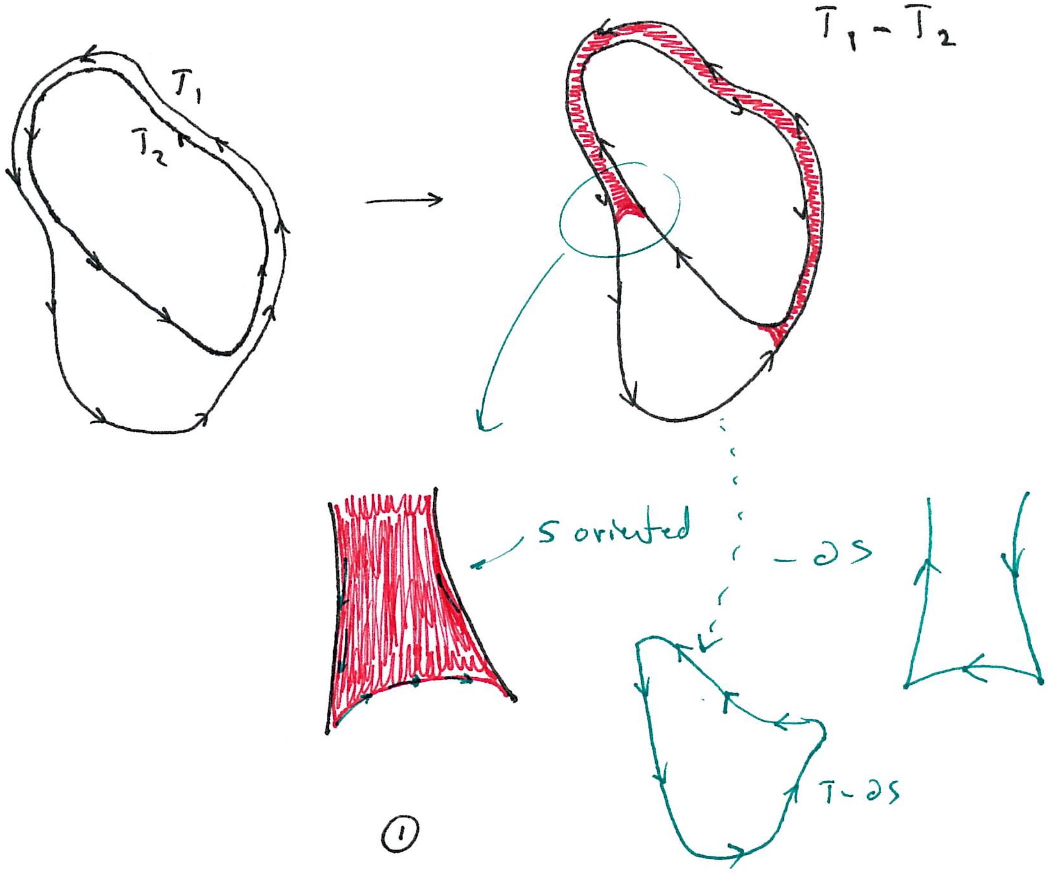
L1TV = Multiscale Flatnorm: computation and applications

In the last lecture, we saw that one measure of boundaries was the Flat Norm:

$$\underbrace{\mathbb{F}(T)}_{\text{n-1 dim current}} \equiv \min_{\underbrace{S}_{\text{n-dim current}}} \underbrace{M(T - \partial S)}_{\text{n-1 dim volume}} + \underbrace{M(S)}_{\text{n dim volume}}$$

For the purposes of this lecture, currents can be seen as manifolds with orientation or unions of pieces of manifolds with orientation. A more careful, complete exposition will be given in the first of the 6 optional lectures on GMT.





Theorem: for

$$\mathcal{F}_\lambda(u) \equiv \int |\nabla u|\,dx + \lambda \int |u-f|\,dx$$

$$\mathbb{F}(T) \equiv \min_S \; M(T-\partial S) + M(S)$$

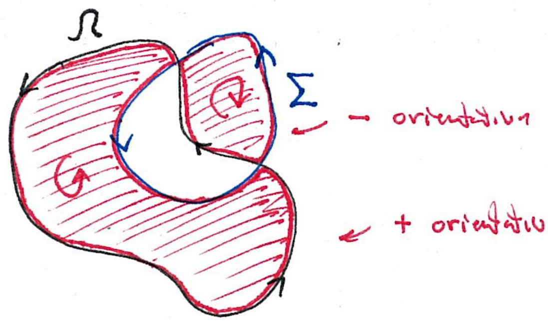
$$\mathcal{F}_1(\chi_\Omega) = \mathbb{F}(\partial\Omega)$$

Proof: A picture, but first some facts

(a) $\int |\nabla \chi_\Sigma| = Per(\Sigma)$

(b) for L1TV χ_Ω in $\longrightarrow$ $\exists\, \Sigma \ni \chi_\Sigma$ is a minimizer

$\Rightarrow$ $L1TV(\chi_\Omega) \Rightarrow Per(\Sigma) + \lambda\,|\Omega \Delta \Sigma|$



① $\partial\Omega - \partial S = \partial\Sigma$

② $M(\partial\Omega - \partial S) = M(\partial\Sigma) = Per(\Sigma)$

③ $M(S) = |\Omega \Delta \Sigma|$

$\mathbb{F}(\partial\Omega) =$

$\Rightarrow$ $M(\partial\Omega - \partial S) + M(S) = Per(\Sigma) + 1\,|\Omega \Delta \Sigma|$

$= \mathcal{F}_1(\chi_\Sigma)$

we immediately get generalizations of both $\mathcal{F}_\lambda(u)$ and $\mathbb{F}(T)$:

(a) For sets that are not boundaries and codimension 1 $\mathbb{F}(T)$ works perfectly well, generalizing $\mathcal{F}_\lambda(\chi_\Omega)$ to these sets.

(b) We can change scales with $\mathcal{F}_\lambda$ by changing λ: this gives a multiscale metric/measure and decomposition.

$$\Rightarrow \quad \mathbb{F}_\lambda(T) \equiv \underbrace{\min_S}_{\text{minimizes } S \to S_\lambda} M(T-\partial S) + \lambda M(S)$$

Decompositions: $\quad T = T - \partial S_\lambda + \partial S_\lambda$

Signatures:

$$f(\lambda) = \mathbb{F}_\lambda(T)$$

$$f_{tmbs.}(\lambda) = M(T-\partial S_\lambda)$$

$$f_S(\lambda) = M(S)$$

(~~Some~~ slides on signatures and shape classifications will be shown later in lecture.)

Facts:

(a) λ is a curvature bound, see Allard's theory paper

(b) if $B_{2/\lambda} \subset \Omega \to B_{2/\lambda} \subset \Sigma$ ← nonuniqueness issue

if $B_{2/\lambda} \subset \Omega^c \to B_{2/\lambda} \subset \Sigma^c$

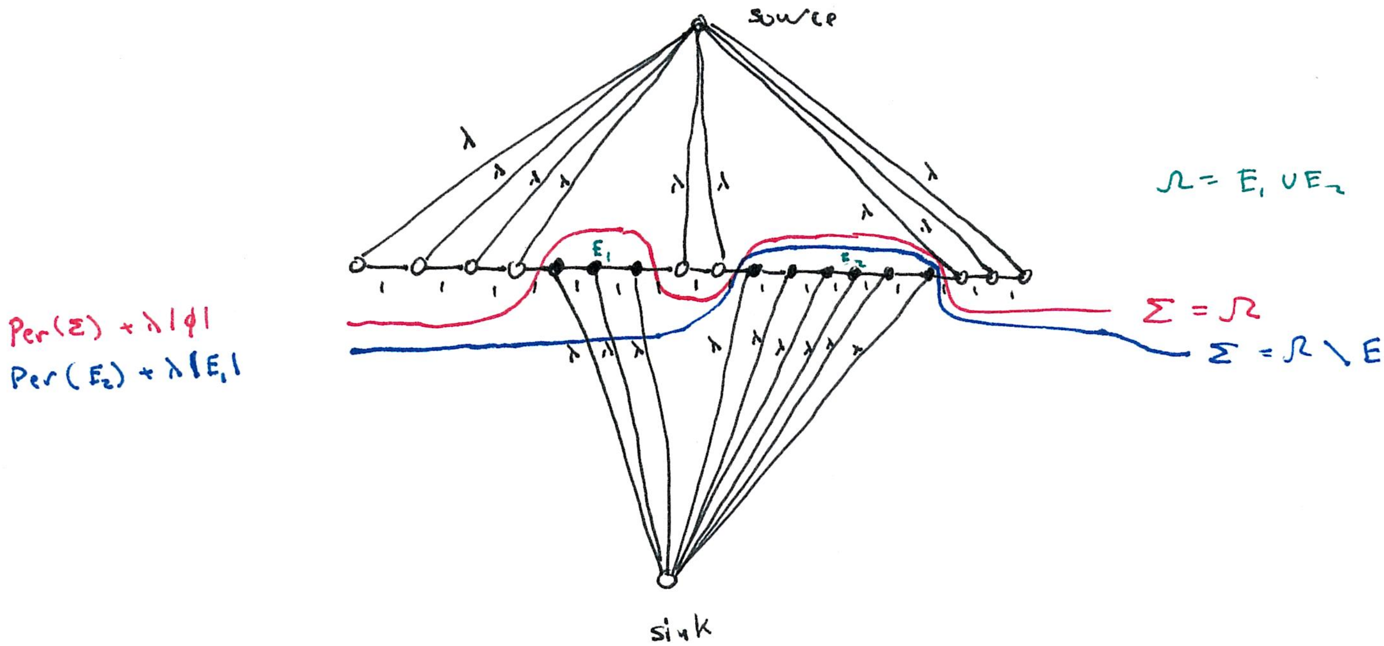
(c) $\partial\Sigma = \partial\Omega$ or $=$ arcs of radius $\frac{1}{\lambda}$ being in and out appropriately

(d) $\partial\Sigma \setminus \partial\Omega$ has arcs of arclength $< \pi$.

Computing LIJV

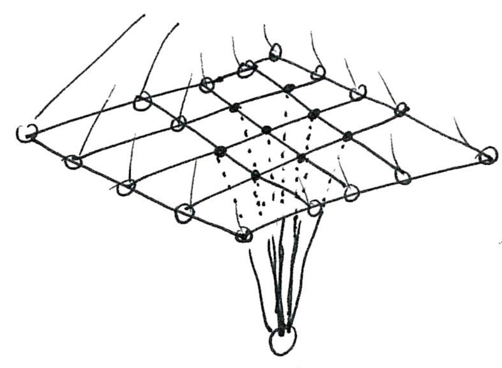
The big deal with this equivalence is that it immediately gave us a computational path to $\mathbb{F}_\lambda(J)$. What we have been using is a code based on Wotao Yin's graph cut algorithm.

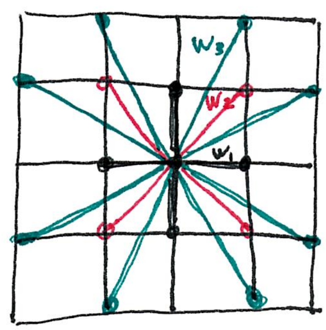
First I will illustrate the approach with a 1-D problem



goal: find a cut separating source from sink which has smallest total weight   The min cut - max flow problem

In $\mathbb{R}^2$, we again hook up Ω^c to source and Ω to sink and pixels to neighbors, more neighbors for more accurate approximations of $\int |\nabla \chi_\Omega|$.





Task: choose weights such that

$$w_1 \left(|f_{i+1,j} - f_{i,j}| + |f_{i,j+1} - f_{i,j}| + |f_{i-1,j} - f_{i,j}| + |f_{i,j-1} - f_{i,j}| \right)$$

$$+ \; w_2 \left(|f_{i+1,j+1} - f_{i,j}| + | \cdots \right)$$

$$+ \; w_3 \left(|f_{i+2,j+1} - f_{i,j}| + \cdots \right)$$

is as close as possible to $|\nabla f_{i,j}|$.

We do this by choosing f to be a plane with gradient vector $\vec{v}$. We do this by minimizing

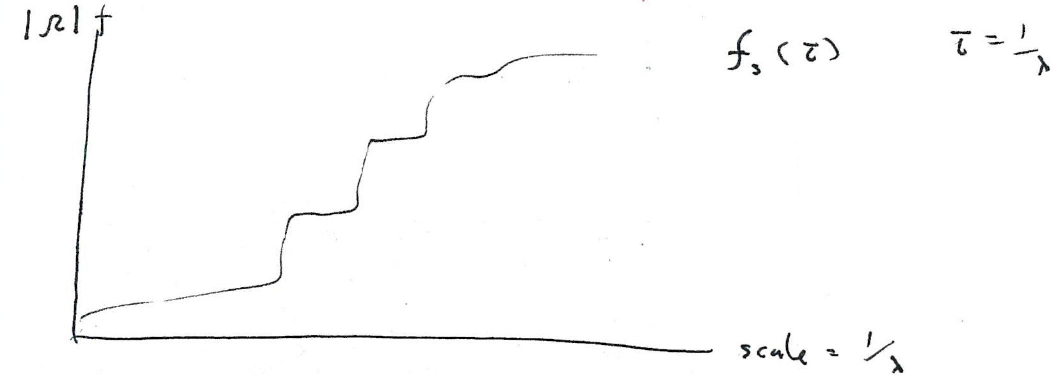
$$\int_0^{2\pi} \left(\sum_{k=1}^{3} w_k \left(\sum_i |\vec{v} \cdot \vec{\Delta}_i| \right) - 1 \right)^2$$

where $\vec{v} \in \partial B(0,1)$.

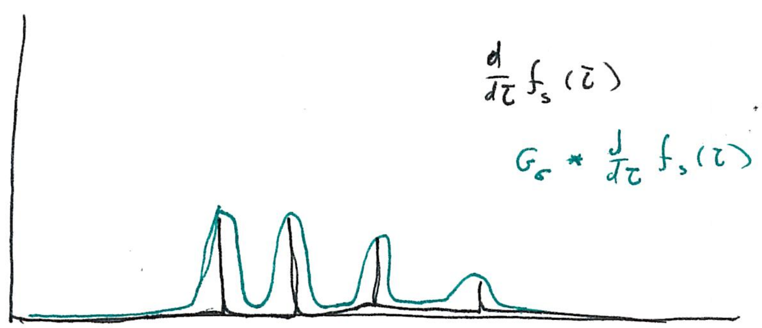
(Croftons formula can also be used to find w_k's but this method is a bit less accurate in getting perimeters right)

The signatures mentioned above are useful in shape recognition tasks.

Example: we took images of "spoons" and "forks" and computed f_s, the area signature.



To focus on the scale information the signatures contain, we differentiate



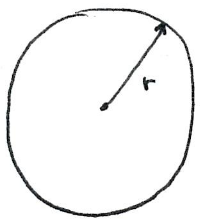
These spectra contained information that made separation of spoons and forks trivial... linear classifiers worked well.

(show slides)

More facts and examples involving L^1 TV.

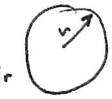
Disks

$f = u_{in} = \chi_{B_r}$



$\Rightarrow$

$$u_{out} = \begin{cases} \chi_{B_r} & \boxed{r > \frac{2}{\lambda}} \\ \alpha \chi_{B_r}, \quad 0 \le \alpha \le 1 & \boxed{r = \frac{2}{\lambda}} \\ \chi_{\emptyset} = 0! & \boxed{r < \frac{2}{\lambda}} \end{cases}$$



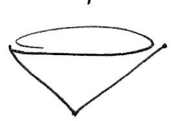
nonuniqueness

notice: minimizing set is convex!

Anisotropy

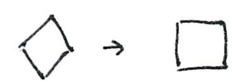
The graphcut approach ends up enforcing an anisotropic perimeter term.

The dual shape (Lengendre - Fenchel) for the ball is the "same" ball ... well, actually the homogeneous function based on the ball



... some details.

For polygons we get different polygons: e.g.



Anyway, the roll played by arcs of circles in the isotropic case, is played by the dual shape (called the wulff shape) in the anisotropic case.

Convex Ω

For convex Ω, if $\Sigma \neq \phi$ then $\Sigma = \bigcup_{B_* \subset \Omega} B_{1/\lambda}$

Examples: (1) $\Omega = B(0, \frac{1.5}{\lambda}) \Rightarrow \Sigma = \phi$

(2)