PCMI Lecture # 11

Geometric Analysis: Intro, perimeters, densities and the coarea formula

This is the first of five lectures on geometric analysis and its applications to shapes and images.

Geometric analysis, which I define as the strong intersection of geometry and analysis as found in geometric measure theory, variational analysis and pieces of harmonic analysis, and PDE, is a very rich area of research, which is significantly underexploited for its usefulness for data and image analysis.

What I will talk about in these five lectures is contained in geometric measure theory, a subject I will delve into more deeply in the 6 optional lectures that will be given later in this summer school.

Geometric measure theory:

Early period 1900 – 1960

- Besicovitch
- De Giorgi
- Federer
- Hausdorff
- ⋮

Modern Period 1960 – present

- Federer & Fleming
- Reifenberg
- De Giorgi

} 1960, Plateau's Problem

- Almgren
- Allard
- Preiss < Ambrosio
- Hardt
- White
- etc.
- Jones
- Mattila
- etc.

2 main branches

Calculus of Variations

$\boxed{\text{minimizers of variational functionals}}$

$$\min_u \int_\Omega |\nabla u|\, dx$$

$$u = f \text{ on } \partial\Omega$$

Harmonic measure
Harmonic Analysis

$\boxed{\text{Harmonic measures}}$

$$\Delta H_f = 0$$

$$H_f = f \text{ on } \partial\Omega$$

$$H_f(x) = \int f(y)\, d\omega(x,\Omega)(y)$$

GMT studies sets, measures and functions with a perspective that integrates geometry and analysis.

We will look at three topics:

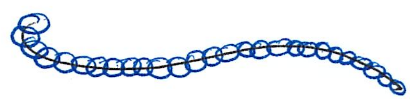
1. How to measure the size of ~~[illegible]~~ boundaries in $\mathbb{R}^n$
2. Boundary signatures, using densities.
3. The coarea formula, a far reaching generalization of Fubini ... with more boundaries thrown in for good measure.

The size of $\partial\Omega : \Omega \subset \mathbb{R}^n$

Reminder: Hausdorff measure

$$\mathcal{H}^{n-1}(\partial\Omega) = \sup_{\varepsilon\to 0} \inf_{\boxed{\mathcal{F}}} \alpha(n-1)\sum_{\substack{i=1 \\ F_i\in\mathcal{F}}}^{\infty} \left(\frac{\text{diam}(F_i)}{2}\right)^{n-1}$$

$$\boxed{\begin{array}{c}\text{diam}\, F < \varepsilon \\ \forall F \in \mathcal{F}\end{array}} \qquad \boxed{\partial\Omega \subset \bigcup_{F\in\mathcal{F}} F}$$



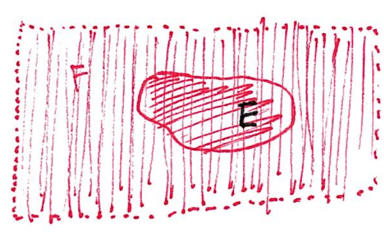
First measure:

$\mathcal{H}^{n-1}(\partial\Omega)$ where $\partial\Omega$ is the topological boundary of Ω:

$x \in \partial\Omega$ if there are elements of Ω and Ω^c in every neighborhood of x.

While there may be some applications for which this is the right measure, it is often not what we want because it allows small bits to dictate large things: Example

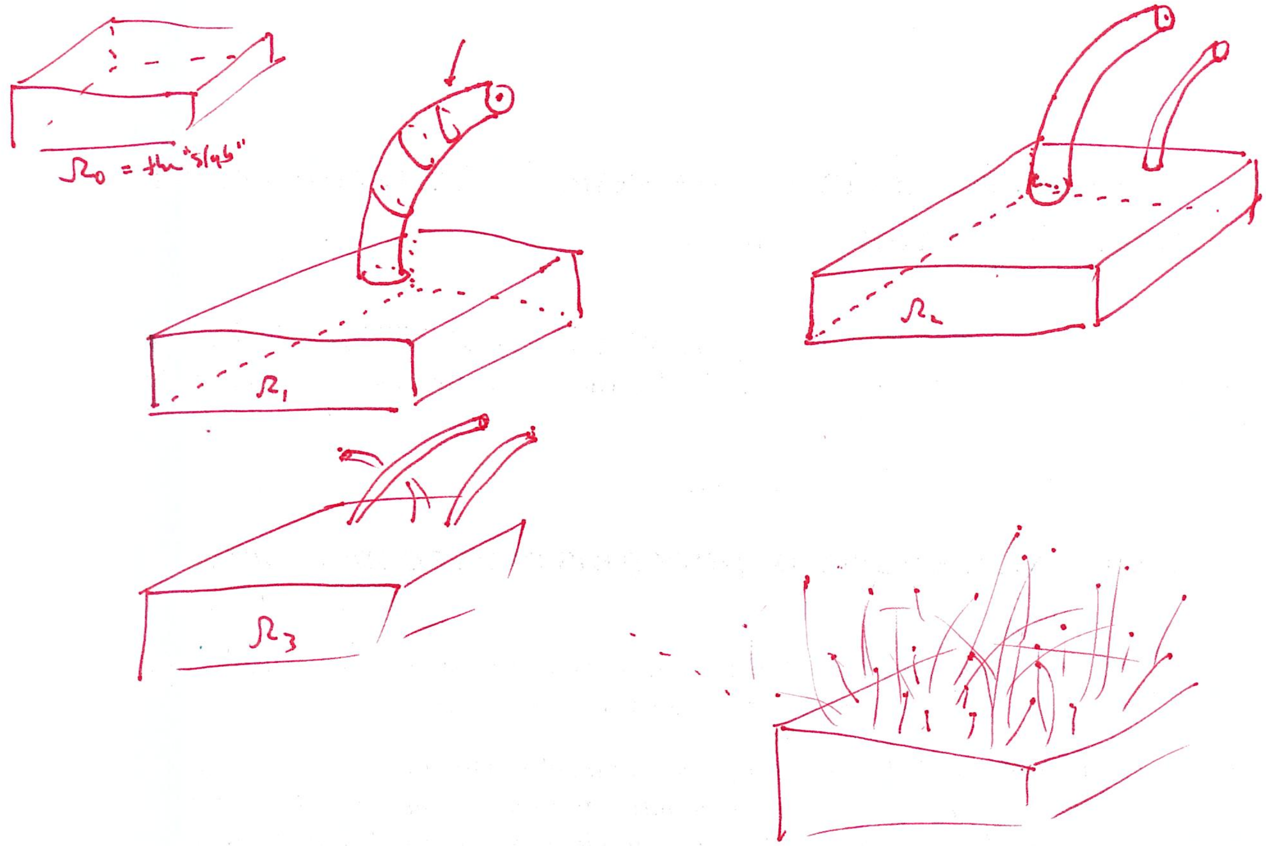
$\Omega = E \cup F$



• vertical lines at rational x coordinates.

$\Rightarrow$ $\partial\Omega$ is everything outside of $\overset{\circ}{E}$, ~~[illegible]~~

In $\mathbb{R}^3$, we can construct an example highlighting why the topological boundary is not what we want for variational studies.



$\mathcal{H}^3(\Omega_i) \leq 1 + \frac{1}{2^i}$ $\quad \mathcal{H}^2(\partial\Omega_i) \leq C$ $\quad$ (or even $f(i) \xrightarrow[i\to\infty]{} \mathcal{H}^2(\Omega_0)$)

$\Omega \equiv \bigcap \Omega_i$ = slab + hair

$\partial\Omega = \mathbb{R}^3 \setminus \Omega_0^\circ$ (interior of slab)

$L^1(\chi_\Omega - \chi_{\Omega_i}) \to 0 \quad i \to \infty$, $\quad \mathcal{H}^2(\partial\Omega_i) \leq C$

and

$\mathcal{H}^2(\partial\Omega) = \infty$

so we lose lower semicontinuity ... ☹ bad news for the direct method of calculus of variations.

i.e. $\quad \Omega_i \to \Omega \quad$ but $\quad \hat{F}(\Omega) > \liminf_{i\to\infty} \hat{F}(\Omega_i)$

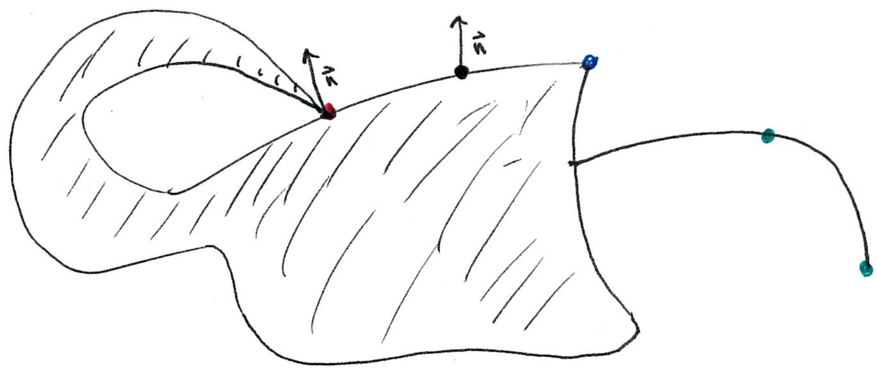
where $\quad \hat{F} \equiv \mathcal{H}^2(\partial\,\cdot)$

Second measure

$$\mathrm{Per}(\Omega) \equiv \int |\nabla \chi_\Omega| \, dx$$

Recall: $\int |\nabla \chi_\Omega| \, dx = \sup_{\phi} \left\{ \int \chi_\Omega (\nabla \cdot \vec{\phi}) \, dx \;\middle|\; \|\vec{\phi}\| = 1 \text{ a.e. } \vec{\phi} \text{ is } C^1 \right\}$

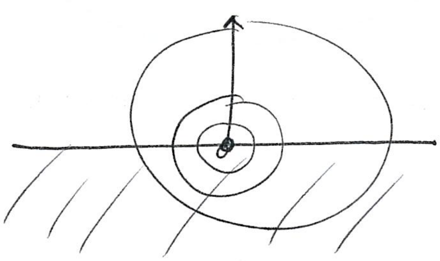
Moral of the story: $\mathrm{Per}(\Omega) = \mathcal{H}^{n-1}(\partial^* \Omega)$, where $\partial^*\Omega$ is the reduced boundary of Ω. What is the reduced boundary?



$\partial \Omega$ • • • •

$\partial^* \Omega$ • •

A point is in the reduced boundary if there is ~~a~~ measure theoretic exterior normal



Third Measure

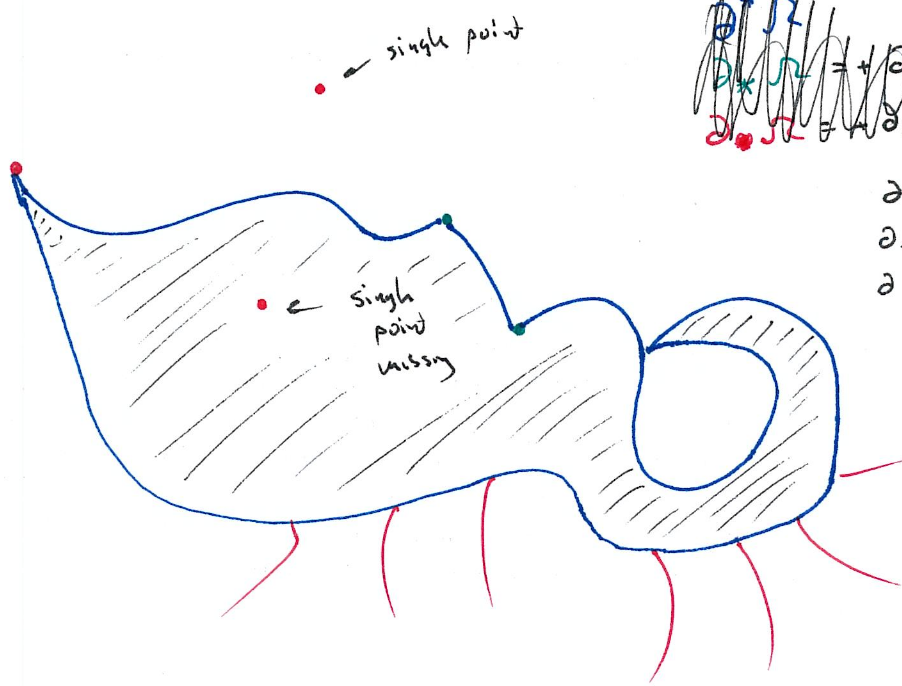
$\mathcal{H}^{n-1}(\partial_* \Omega)$: measure theoretic boundary

$$x \in \partial_* \Omega \text{ if } \begin{cases} \limsup\limits_{r \to 0} \dfrac{\mathcal{L}^n(B_r(x) \cap E)}{r^n} > 0 \\ \\ \limsup\limits_{r \to 0} \dfrac{\mathcal{L}^n(B_r(x) \setminus E)}{r^n} > 0 \end{cases}$$

① $\partial^* \Omega \subset \partial_* \Omega$

$\Rightarrow$ ② $\mathcal{H}^{n-1}(\partial_* \Omega - \partial^* \Omega) = 0 \Leftarrow$

While in some ways $\partial_* \Omega$ is easier to think about, it is easier to compute $\mathcal{H}^{n-1}(\partial^* \Omega)$ $(= \mathcal{H}^{n-1}(\partial_* \Omega).)$

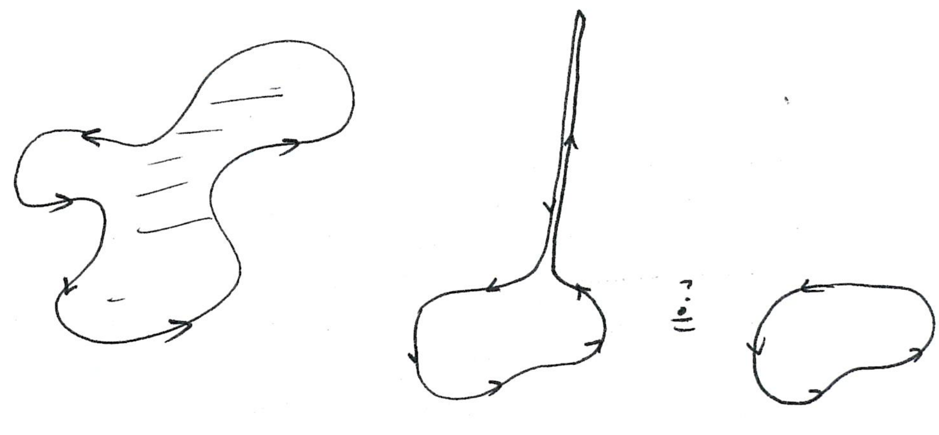


$\partial^* \Omega = \bullet$

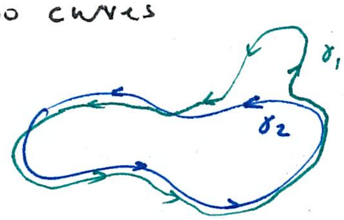
$\partial_* \Omega = \bullet + \bullet$

$\partial \Omega = \bullet + \bullet + \bullet$

But... suppose $\partial\Omega$ is oriented

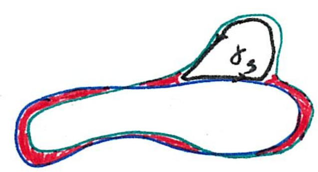


that is, suppose we want to measure the size of a difference between two curves



clearly, $\mathcal{H}^1(\bullet - \bullet) = \mathcal{H}^1(\bullet) + \mathcal{H}^1(\bullet)$

so we allow a decomposition



$$\gamma_1 + \gamma_2 = \gamma_3 + \partial S$$

Define $\quad \mathbb{F}(\gamma_1 + \gamma_2) = \min_S \left(\mathcal{H}^1(\gamma_1 + \gamma_2 - \partial S) + \mathcal{H}^2(S) \right)$

comment: This definition works for any orientable "curve", surface in any codimension ... for any current.

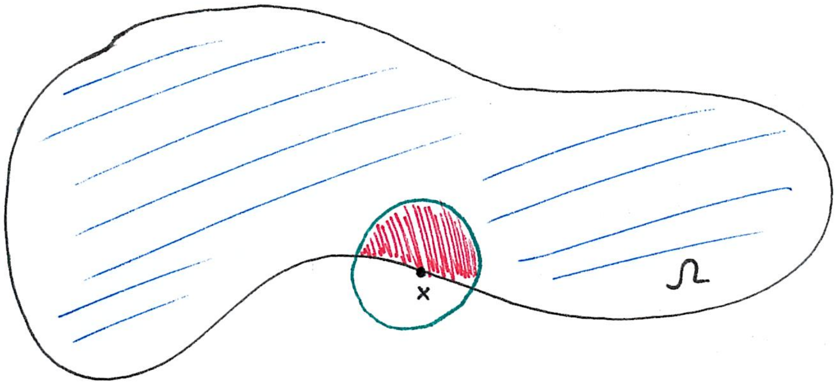
Fourth Measure

boundary of Ω as a current

$$\mathbb{F}(\partial\Omega)$$

Boundary Signatures from Densities

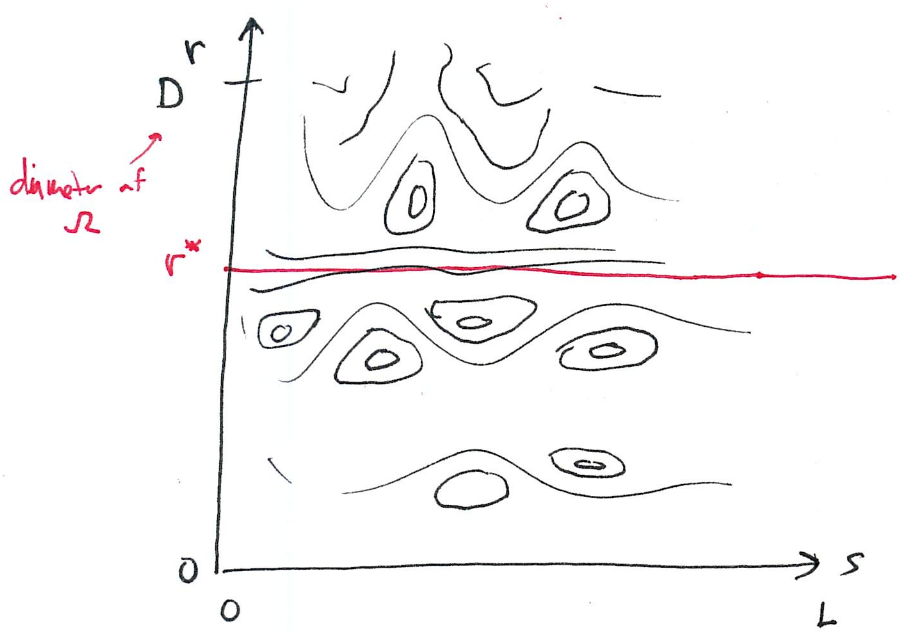
$\mathbb{R}^2$



$$\lim_{r \to 0} \frac{\mathcal{L}^2(B_r(x) \cap \Omega)}{\alpha(n)\, r^2}$$

($\left.\begin{matrix}\limsup \\ \liminf\end{matrix}\right\} \dfrac{\mathcal{L}^n(B_r(x) \cap \Omega)}{\alpha(n)\, r^n}$ used all the time in geometric analysis)

Parameterizing $\partial\Omega$ by arclength to get $\partial\Omega = \{\bigcup_s \gamma(s) \mid 0 \le s \le L\}$ we get a 2 dimensional signature for a simple closed curve.



signature of $\partial\Omega$, $g(s)$

Q: Can we reconstruct Ω from $g(s)$?

A: Yes, given information along r^* we can reconstruct a dense set of Ω's... that is what we can prove, but we think something much stronger is true

Bonus: $g(s)$ is useful!!

Coarea Formula

In Federer's paper from 1959 titled "Curvature Measures", he introduced what has now become known as the coarea formula.

$f : \mathbb{R}^n \to \mathbb{R}^m \qquad n \geq m \quad , f$ is Lipschitz

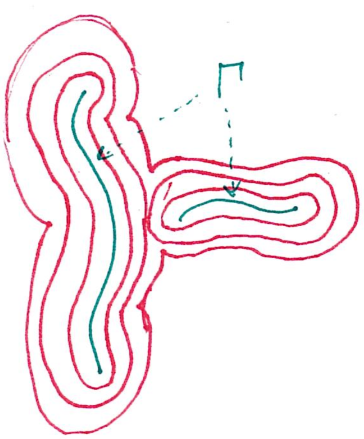
$Jf = \sqrt{\det(Df \circ Df^*)}$

$$(1) \qquad \int_A Jf \, dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \, dy$$

$$(2) \qquad \int_A g(x) \, Jf \, dx = \int_{\mathbb{R}^m} \int_{A \cap f^{-1}(\{y\})} g(x) \, d\mathcal{H}^{n-m}(x) \, dy$$

The proof is rather involved

comment: this generalizes Fubini - choosing f to be a distance function from some measure zero set, say a curve or collection of curves Γ, we get



$f : \mathbb{R}^2 \to \mathbb{R}^1$, we can integrate g over level sets of the distance to Γ, and then integrate these versus the distance to get $\int_{\mathbb{R}^2} g \, dx$