Introduction to Geometric Measure Theory

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Abstract

These are the notes to four one-hour lectures I delivered at the spring school "Geometric Measure Theory: Old and New" which took place in Les Diablerets, Switzerland, from April 3–8, 2005 (see `http://igat.epfl.ch/diablerets05/`). The first three of these lectures were intended to provide the fundamentals of the "old" theory of rectifiable sets and currents in euclidean space as developed by Besicovitch, Federer–Fleming, and others. The fourth lecture, independent of the previous ones, discussed some metric space techniques that are useful in connection with the new metric approach to currents by Ambrosio–Kirchheim. Other short courses were given by G. Alberti, M. Csörnyei, B. Kirchheim, H. Pajot, and M. Zähle.

Contents

Lecture 1: Rectifiability

Lipschitz maps

Let X, Y be metric spaces, and let $\lambda \in [0, \infty)$. A map $f\colon X \to Y$ is λ-*Lipschitz* if

$$d(f(x), f(x')) \le \lambda\, d(x, x') \quad \text{for all } x, x' \in X;$$

f is *Lipschitz* if

$$\mathrm{Lip}(f) := \inf\{\lambda \in [0, \infty)\colon f \text{ is } \lambda\text{-Lipschitz}\} < \infty.$$

The following basic extension result holds, see [McS] and the footnote in [Whit].

1.1 Lemma (McShane, Whitney)
Suppose X is a metric space and $A \subset X$.
(1) *For $n \in \mathbb{N}$, every λ-Lipschitz map $f\colon A \to \mathbb{R}^n$ admits a $\sqrt{n}\lambda$-Lipschitz extension $\bar{f}\colon X \to \mathbb{R}^n$.*
(2) *For any set J, every λ-Lipschitz map $f\colon A \to l^\infty(J)$ has a λ-Lipschitz extension $\bar{f}\colon X \to l^\infty(J)$.*

Proof: (1) For $n = 1$, put

$$\bar{f}(x) := \inf\{f(a) + \lambda\, d(a, x)\colon a \in A\}.$$

For $n \ge 2$, $f = (f_1, \dots, f_n)$, extend each f_i separately.

(2) For $f = (f_j)_{j \in J}$, extend each f_j separately. □

In (1), the factor $\sqrt{n}$ cannot be replaced by a constant $< n^{1/4}$, cf. [JohLS] and [Lan]. In particular, Lipschitz maps into a Hilbert space Y cannot be extended in general. However, if X is itself a Hilbert space, one has again an optimal result:

1.2 Theorem (Kirszbraun, Valentine)
If X, Y are Hilbert spaces, $A \subset X$, and $f\colon A \to Y$ is λ-Lipschitz, then f has a λ-Lipschitz extension $\bar{f}\colon X \to Y$.

See [Kirs], [Val], or [Fed, 2.10.43]. A generalization to metric spaces with curvature bounds was given in [LanS].

The next result characterizes the extendability of partially defined Lipschitz maps from $\mathbb{R}^m$ into a complete metric space Y; it is useful in connection with the definition of rectifiable sets (Def. 1.13). We call a metric space Y *Lipschitz m-connected* if there is a constant $c \ge 1$ such that for $k \in \{0, \dots, m\}$, every λ-Lipschitz map $f\colon \mathrm{S}^k \to Y$ admits a $c\lambda$-Lipschitz extension $\bar{f}\colon \mathrm{B}^{k+1} \to Y$; here S^k and B^{k+1} denote the unit sphere and closed ball in $\mathbb{R}^{k+1}$, endowed with the induced metric. Every Banach space is Lipschitz m-connected for all $m \geq 0$. The sphere S^n is Lipschitz $(n-1)$-connected.

1.3 Theorem (Lipschitz maps on $\mathbb{R}^m$)
Let Y be a complete metric space, and let $m \in \mathbb{N}$. Then the following statements are equivalent:
(1) *Y is Lipschitz $(m-1)$-connected.*
(2) *There is a constant c such that every λ-Lipschitz map $f \colon A \to Y$, $A \subset \mathbb{R}^m$, has a $c\lambda$-Lipschitz extension $\bar{f} \colon \mathbb{R}^m \to Y$.*

The idea of the proof goes back to Whitney [Whit]. Compare [Alm1, Thm. (1.2)] and [JohLS].

Proof: It is clear that (2) implies (1). Now suppose that (1) holds, and let $f \colon A \to Y$ be a λ-Lipschitz map, $A \subset \mathbb{R}^m$. As Y is complete, assume w.l.o.g. that A is closed. A *dyadic cube* in $\mathbb{R}^m$ is of the form $x + [0, 2^k]^m$ for some $k \in \mathbb{Z}$ and $x \in (2^k\mathbb{Z})^m$. Denote by $\mathcal{C}$ the family of all dyadic cubes $C \subset \mathbb{R}^m \setminus A$ that are maximal (with respect to inclusion) subject to the condition

$$\operatorname{diam} C \leq 2\, d(A, C).$$

They have pairwise disjoint interiors, cover $\mathbb{R}^m \setminus A$, and satisfy

$$d(A, C) < 2 \operatorname{diam} C,$$

for otherwise the next bigger dyadic cube C' containing C would still fulfill

$$\operatorname{diam} C' = 2 \operatorname{diam} C \leq 2(d(A, C) - \operatorname{diam} C) \leq 2\, d(A, C').$$

Denote by $\Sigma_k \subset \mathbb{R}^m$ the k-skeleton of this cubical decomposition. Extend f to a Lipschitz map $f_0 \colon A \cup \Sigma_0 \to Y$ by precomposing f with a nearest point retraction $A \cup \Sigma_0 \to A$. Then, for $k = 0, \ldots, m-1$, successively extend f_k to $f_{k+1} \colon A \cup \Sigma_{k+1} \to Y$ by means of the Lipschitz $(m-1)$-connectedness of Y. As $A \cup \Sigma_m = \mathbb{R}^m$, $\bar{f} := f_m$ is the desired extension of f. □

Differentiability

Recall the following definitions.

1.4 Definition (Gâteaux and Fréchet differential)
Suppose X, Y are Banach spaces, f maps an open set $U \subset X$ into Y, and $x \in U$.
(1) The map f is *Gâteaux differentiable* at x if the directional derivative $D_v f(x)$ exists for every $v \in X$ and if there is a continuous linear map $L \colon X \to Y$ such that

$$L(v) = D_v f(x) \quad \text{for all } v \in X.$$

Then L is the *Gâteaux differential* of f at x.

(2) The map f is (*Fréchet*) *differentiable* at x if there is a continuous linear map $L\colon X \to Y$ such that

$$\lim_{v\to 0} \frac{f(x+v) - f(x) - L(v)}{\|v\|} = 0.$$

Then $L =: Df_x$ is the (*Fréchet*) *differential* of f at x.

The map f is Fréchet differentiable at x iff f is Gâteaux differentiable at x and the limit in

$$L(u) = \lim_{t\to 0}(f(x+tu) - f(x))/t$$

exists uniformly for u in the unit sphere of X, i.e. for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$\|f(x+tu) - f(x) - tL(u)\| \le \epsilon|t|$$

whenever $|t| \le \delta$ and $u \in \mathrm{S}(0,1) \subset X$.

1.5 Lemma (differentiable Lipschitz maps)
Suppose Y is a Banach space, $f\colon \mathbb{R}^m \to Y$ is Lipschitz, $x \in \mathbb{R}^m$, D is a dense subset of S^{m-1}, $D_u f(x)$ exists for every $u \in D$, $L\colon \mathbb{R}^m \to Y$ is linear, and $L(u) = D_u f(x)$ for all $u \in D$. Then f is Fréchet differentiable at x with differential $Df_x = L$.

In particular, if $f\colon \mathbb{R}^m \to Y$ is Lipschitz and Gâteaux differentiable at x, then f is Fréchet differentiable at x.

Proof: Let $\epsilon > 0$. Choose a finite set $D' \subset D$ such that for every $u \in \mathrm{S}^{m-1}$ there is a $u' \in D'$ with $|u - u'| \le \epsilon$. Then there is a $\delta > 0$ such that

$$\|f(x+tu') - f(x) - tL(u')\| \le \epsilon|t|$$

whenever $|t| \le \delta$ and $u' \in D'$. Given $u \in \mathrm{S}^{m-1}$, pick $u' \in D'$ with $|u-u'| \le \epsilon$; then

$$\begin{aligned} &\|f(x+tu) - f(x) - tL(u)\| \\ &\quad \le \epsilon|t| + \|f(x+tu) - f(x+tu')\| + |t|\|L(u-u')\| \\ &\quad \le (1 + \mathrm{Lip}(f) + \|L\|)\epsilon|t| \end{aligned}$$

for all $|t| \le \delta$. □

1.6 Theorem (Rademacher)
Every Lipschitz map $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\mathcal{L}^m$-almost all points in $\mathbb{R}^m$.

This was originally proved in [Rad].

Proof: It suffices to prove the theorem for $n = 1$; in the general case, $f = (f_1, \dots, f_n)$ is differentiable at x iff each f_i is differentiable at x.

In the case $m = 1$ the function $f\colon \mathbb{R} \to \mathbb{R}$ is absolutely continuous and hence $\mathcal{L}^1$-almost everywhere differentiable.

Now let $m \geq 2$. For $u \in \mathrm{S}^{m-1}$, denote by B_u the set of all $x \in \mathbb{R}^m$ where $D_u f(x)$ exists and by H_u the linear hyperplane orthogonal to u. For $x_0 \in H_u$, the function $t \mapsto f(x_0 + tu)$ is $\mathcal{L}^1$-almost everywhere differentiable by the result for $m = 1$, hence

$$\mathcal{H}^1((x_0 + \mathbb{R}u) \setminus B_u) = 0.$$

Since B_u is a Borel set, Fubini's theorem implies

$$\mathcal{L}^m(\mathbb{R}^m \setminus B_u) = 0.$$

Now choose a dense countable subset D of S^{m-1}. Then it follows that for $\mathcal{L}^m$-almost every $x \in \mathbb{R}^m$, $D_u f(x)$ and $D_{e_i} f(x)$ exist for all $u \in D$ and $i = 1, \dots, m$; in particular, the formal gradient

$$\nabla f(x) := (D_{e_i} f(x), \dots, D_{e_m} f(x))$$

exists. We show that for $\mathcal{L}^m$-almost all $x \in \mathbb{R}^m$ we have, in addition, the usual relation

$$D_u f(x) = \langle \nabla f(x), u \rangle \quad \text{for all } u \in D.$$

The theorem then follows from Lemma 1.5. Let $\varphi \in C_c^\infty(\mathbb{R}^m)$. By Lebesgue's bounded convergence theorem,

$$\lim_{t \to 0+} \int \frac{f(x + tu) - f(x)}{t} \varphi(x)\, dx = \int D_u f(x) \varphi(x)\, dx,$$
$$\lim_{t \to 0+} \int f(x) \frac{\varphi(x - tu) - \varphi(x)}{t}\, dx = -\int f(x) D_u \varphi(x)\, dx.$$

Substituting $x + tu$ by x in the term $f(x + tu)\varphi(x)$ we see that the two left-hand sides coincide. Hence,

$$\int D_u f(x) \varphi(x)\, dx = -\int f(x) D_u \varphi(x)\, dx,$$

and similarly

$$\int \langle \nabla f(x), u \rangle \varphi(x)\, dx = -\int f(x) \langle \nabla \varphi(x), u \rangle\, dx.$$

Now the right-hand sides of these two identities coincide. As $\varphi \in C_c^\infty(\mathbb{R}^m)$ is arbitrary, we conclude that $D_u f(x) = \langle \nabla f(x), u \rangle$ for $\mathcal{L}^m$-almost every $x \in \mathbb{R}^m$. $\square$

1.7 Theorem (Stepanov)
Every function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\mathcal{L}^m$-almost all points in the set
$$L(f) := \bigl\{x\colon \limsup_{y\to x} |f(y) - f(x)|/|y - x| < \infty\bigr\}.$$

This generalization of Rademacher's theorem was proved in [Ste]. The following elegant argument is due to Malý [Mal].

Proof: It suffices to consider the case $n = 1$. Let $(U_i)_{i\in\mathbb{N}}$ be the family of all open balls in $\mathbb{R}^m$ with rational center and radius such that $f|U_i$ is bounded. This family covers $L(f)$. Let $a_i\colon U_i \to \mathbb{R}$ be the supremum of all i-Lipschitz functions $\le f|U_i$, and let $b_i\colon U_i \to \mathbb{R}$ be the infimum of all i-Lipschitz functions $\ge f|U_i$. Note that a_i, b_i are i-Lipschitz and $a_i \le f|U_i \le b_i$. Let

$$A_i := \{x \in U_i\colon \text{both } a_i \text{ and } b_i \text{ are differentiable at } x\}.$$

By Rademacher's theorem, $Z := \bigcup_{i=1}^{\infty} U_i \setminus A_i$ has measure zero. Let $x \in L(f) \setminus Z$. We show that for some i, $x \in A_i$ and $a_i(x) = b_i(x)$; then f is differentiable at x. Since $x \in L(f)$, there is a radius $r > 0$ such that $|f(y) - f(x)| \le \lambda|y - x|$ for all $y \in \mathrm{B}(x, r)$ and for some λ independent of y. Choose i such that $i \ge \lambda$ and $x \in U_i \subset \mathrm{B}(x, r)$. Since $x \notin Z$, $x \in A_i$. By the definition of a_i and b_i,

$$f(x) - i|y - x| \le a_i(y) \le f(y) \le b_i(y) \le f(x) + i|y - x|$$

for all $y \in U_i$. Hence, $a_i(x) = b_i(x)$. □

Generalizations of these results to maps between Banach spaces or even more general classes of metric spaces are a topic of current research.

Finally, we state Whitney's extension theorem for C^1 functions and an application, cf. [Whit], [Fed, 3.1.14] and [Sim, 5.3], [Fed, 3.1.16].

1.8 Theorem (Whitney)
Suppose $f\colon A \to \mathbb{R}$ is a function on a closed set $A \subset \mathbb{R}^m$, $g\colon A \to \mathbb{R}^m$ is continuous, and for all compact sets $C \subset A$ and all $\epsilon > 0$ there is a $\delta > 0$ such that
$$|f(y) - f(x) - \langle g(x), y - x\rangle| \le \epsilon|y - x|$$
whenever $x, y \in C$ and $|y-x| \le \delta$. Then there exists a C^1 function $\bar{f}\colon \mathbb{R}^m \to \mathbb{R}$ with $\bar{f}|A = f$ and $\nabla\bar{f}|A = g$.

1.9 Theorem (C^1 approximation of Lipschitz functions)
If $f\colon \mathbb{R}^m \to \mathbb{R}$ is Lipschitz and $\epsilon > 0$, then there is a C^1 function $\bar{f}\colon \mathbb{R}^m \to \mathbb{R}$ such that
$$\mathcal{L}^m(\{x \in \mathbb{R}^m\colon f(x) \ne \bar{f}(x)\}) < \epsilon.$$

Proof: By Rademacher's theorem, f is almost everywhere differentiable, and $g := \nabla f$ is a measurable function. According to Lusin's theorem, there is a closed set $B \subset \mathbb{R}^m$ with $\mathcal{L}^m(\mathbb{R}^m \setminus B) < \epsilon/2$ such that $g|B$ is continuous. For $x \in B$ and $i \in \mathbb{N}$, let

$$r_i(x) := \sup |f(y) - f(x) - \langle g(x), y - x\rangle| \,/\, |y - x|,$$

the supremum taken over all $y \in B$ with $0 < |y - x| \le 1/i$. We know that $r_i \to 0$ pointwise on B as $i \to \infty$. By Egorov's theorem, there is a closed set $A \subset B$ with $\mathcal{L}^m(B \setminus A) < \epsilon/2$ such that $r_i \to 0$ uniformly on compact subsets of A. Now extend $f|A$ to $\mathbb{R}^m$ by means of 1.8. □

Area formula

The next goal is to prove Theorem 1.12 below. We start with a technical lemma, cf. [Fed, 3.2.2], [EvaG, p. 94].

1.10 Lemma (Borel partition)
Suppose $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz, and B is the set of all x where Df_x exists and has rank m. Then for every $\lambda > 1$ there exist a Borel partition $(B_i)_{i\in\mathbb{N}}$ of B and a sequence of euclidean norms $\|\cdot\|_i$ on $\mathbb{R}^m$ (i.e. $\|\cdot\|_i$ is induced by an inner product), such that

$$\lambda^{-1}\|v\|_i \le |Df_x(v)| \le \lambda\|v\|_i,$$
$$\lambda^{-1}\|y - x\|_i \le |f(y) - f(x)| \le \lambda\|y - x\|_i$$

for all $x, y \in B_i$ and $v \in \mathbb{R}^m$.

Proof: Choose a sequence of euclidean norms $\|\cdot\|_j$ on $\mathbb{R}^m$ such that for every euclidean norm $\|\cdot\|$ on $\mathbb{R}^m$ and for every $\epsilon > 0$ there is a $j \in \mathbb{N}$ with

$$(1 - \epsilon)\|v\|_j \le \|v\| \le (1 + \epsilon)\|v\|_j \quad \text{for all } v \in \mathbb{R}^m.$$

Given $\lambda > 1$, pick $\delta > 0$ such that $\lambda^{-1} + \delta < 1 < \lambda - \delta$. For $j, k \in \mathbb{N}$, denote by B_{jk} the Borel set of all $x \in B$ with

(i) $(\lambda^{-1} + \delta)\|v\|_j \le |Df_x(v)| \le (\lambda - \delta)\|v\|_j$ for $v \in \mathbb{R}^m$,

(ii) $|f(x + v) - f(x) - Df_x(v)| \le \delta\|v\|_j$ for $|v| \le 1/k$.

To see that the B_{jk} cover B, let $x \in B$, choose $j \in \mathbb{N}$ such that (i) holds, let $c_j > 0$ be such that $|v| \le c_j\|v\|_j$ for all $v \in \mathbb{R}^m$, and pick $k \in \mathbb{N}$ such that

$$|f(x + v) - f(x) - Df_x(v)| \le (\delta/c_j)|v|$$

whenever $|v| \le 1/k$; then $x \in B_{jk}$. Now if $C \subset B_{jk}$ is a set with $\operatorname{diam} C \le 1/k$, then

$$|f(x + v) - f(x)| \le |Df_x(v)| + \delta\|v\|_j \le \lambda\|v\|_j,$$
$$|f(x + v) - f(x)| \ge |Df_x(v)| - \delta\|v\|_j \ge \lambda^{-1}\|v\|_j$$

whenever $x, x+v \in C$. By subdividing and relabeling the sets B_{jk} appropriately we obtain the result. $\square$

1.11 Definition (jacobian)
Let $L\colon X \to Y$ be a linear map between two inner product spaces, where $\dim X = m$. The m-dimensional *jacobian* $\mathbf{J}_m(L)$ of L is the number satisfying

$$\mathbf{J}_m(L) = \mathcal{H}^m(L(A))/\mathcal{H}^m(A) = \sqrt{\det(L^* \circ L)}$$

for all $A \subset X$ with $\mathcal{H}^m(A) > 0$, where $L^*\colon Y \to X$ is the adjoint map.

If $\|\cdot\|$ is a euclidean norm on $\mathbb{R}^m$, we write $\mathbf{J}_m(\|\cdot\|)$ for $\mathbf{J}_m(L)$ where $L\colon \mathbb{R}^m \to (\mathbb{R}^m, \|\cdot\|)$ is the identity map.

1.12 Theorem (area formula)
Suppose $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz with $m \le n$.
(1) *If $A \subset \mathbb{R}^m$ is $\mathcal{L}^m$-measurable, then*

$$\int_A \mathbf{J}_m(Df_x)\,dx = \int_{\mathbb{R}^n} \#(f^{-1}\{y\} \cap A)\,d\mathcal{H}^m(y).$$

(2) *If u is an $\mathcal{L}^m$-integrable function, then*

$$\int_{\mathbb{R}^m} u(x)\mathbf{J}_m(Df_x)\,dx = \int_{\mathbb{R}^n} \sum_{x \in f^{-1}\{y\}} u(x)\,d\mathcal{H}^m(y).$$

Cf. [Fed, 3.2.3], [EvaG, Sect. 3.3]. The formula says, in particular, that the differential geometric volume of an injective C^1 map $f\colon U \to \mathbb{R}^n$, U an open subset of $\mathbb{R}^m$, equals $\mathcal{H}^m(f(U))$. For a metric space version, see [Kir].

Proof: (1) We assume w.l.o.g. that $\mathcal{L}^m(A) < \infty$.

Case 1: $A \subset \{x\colon Df_x \text{ exists and has rank } m\}$. Let $\lambda > 1$. Using Lemma 1.10 we find a measurable partition $(A_i)_{i\in\mathbb{N}}$ of A and a sequence of euclidean norms $\|\cdot\|_i$ on $\mathbb{R}^m$ such that $f|A_i$ is injective,

$$\lambda^{-m}\mathcal{H}^m_{\|\cdot\|_i}(A_i) \le \mathcal{H}^m(f(A_i)) \le \lambda^m \mathcal{H}^m_{\|\cdot\|_i}(A_i),$$

and $\lambda^{-1}\|\cdot\|_i \le |Df_x(\cdot)| \le \lambda\|\cdot\|_i$ for all $x \in A_i$. This last assertion yields $\lambda^{-m}\mathbf{J}_m(\|\cdot\|_i) \le \mathbf{J}_m(Df_x) \le \lambda^m \mathbf{J}_m(\|\cdot\|_i)$. We conclude that

$$\begin{aligned}\mathcal{H}^m(f(A_i)) &\le \lambda^m \mathcal{H}^m_{\|\cdot\|_i}(A_i) = \lambda^m \mathbf{J}_m(\|\cdot\|_i)\mathcal{L}^m(A_i)\\ &\le \lambda^{2m}\int_{A_i} \mathbf{J}_m(Df_x)\,dx.\end{aligned}$$

Since each $f|A_i$ is injective, summation over i gives

$$\int_{\mathbb{R}^n} \#(f^{-1}\{y\} \cap A)\,d\mathcal{H}^m(y) \le \lambda^{2m}\int_A \mathbf{J}_m(Df_x)\,dx.$$

Similarly,

$$\lambda^{-2m} \int_A \mathbf{J}_m(Df_x)\,dx \le \int_{\mathbb{R}^n} \#(f^{-1}\{y\} \cap A)\,d\mathcal{H}^m(y).$$

As this holds for all $\lambda > 1$, the two integrals are equal.

CASE 2: $A \subset \{x \colon Df_x \text{ exists and has rank } < m\}$. Then $\mathbf{J}_m(Df_x) = 0$ for all $x \in A$. For $\epsilon > 0$, consider the map $F \colon \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$, $F(x) = (f(x), \epsilon x)$. For $x \in A$, it follows that $\|DF_x\| \le \mathrm{Lip}(f) + \epsilon$ and

$$\mathbf{J}_m(DF_x) \le \epsilon(\mathrm{Lip}(f) + \epsilon)^{m-1}.$$

Applying the result of the first case to F, we get

$$\mathcal{H}^m(F(A)) = \int_A \mathbf{J}_m(DF_x)\,dx \le \epsilon(\mathrm{Lip}(f) + \epsilon)^{m-1}\mathcal{L}^m(A).$$

Since $\mathcal{H}^m(f(A)) \le \mathcal{H}^m(F(A))$ for all $\epsilon > 0$, it follows that $\mathcal{H}^m(f(A)) = 0$. Thus both integrals equal 0.

CASE 3: $A \subset \{x \colon Df_x \text{ does not exist}\}$. Then

$$\mathcal{H}^m(f(A)) \le \mathrm{Lip}(f)^m \mathcal{H}^m(A) = \mathrm{Lip}(f)^m \mathcal{L}^m(A) = 0$$

by Rademacher's theorem. Thus both integrals equal 0.

(2) follows from (1) by approximation. □

Rectifiable sets

The following notion is fundamental in geometric measure theory.

1.13 Definition (countably rectifiable set)
Let Y be a metric space. A set $E \subset Y$ is *countably $\mathcal{H}^m$-rectifiable* if there is a sequence of Lipschitz maps $f_i \colon A_i \to Y$, $A_i \subset \mathbb{R}^m$, such that

$$\mathcal{H}^m\big(E \setminus \textstyle\bigcup_i f_i(A_i)\big) = 0.$$

It is often possible to take w.l.o.g. $A_i = \mathbb{R}^m$, e.g. if Y is a Banach space (recall Theorem 1.3).

1.14 Theorem (countably rectifiable sets in $\mathbb{R}^n$)
A set $E \subset \mathbb{R}^n$ is countably $\mathcal{H}^m$-rectifiable if and only if there exists a sequence of m-dimensional C^1 submanifolds M_k of $\mathbb{R}^n$ such that

$$\mathcal{H}^m\big(E \setminus \textstyle\bigcup_k M_k\big) = 0.$$

See [Fed, 3.2.29], [Sim, 11.1].

Proof: Suppose that $\mathcal{H}^m(E \setminus \bigcup_i f_i(\mathbb{R}^m)) = 0$ for a sequence of Lipschitz maps $f_i\colon \mathbb{R}^m \to \mathbb{R}^n$. By Theorem 1.9, we assume w.l.o.g. that the f_i are C^1. Let $U_i \subset \mathbb{R}^m$ be the set of all $x \in \mathbb{R}^m$ where Df_x has rank m. By the area formula, $\mathcal{H}^m(f_i(\mathbb{R}^m \setminus U_i)) = 0$. Hence, $\mathcal{H}^m(E \setminus \bigcup_i f_i(U_i)) = 0$. Finally, it follows from the inverse function theorem that each $f_i(U_i)$ is a countable union of C^1 submanifolds.

The other implication is clear. □

1.15 Theorem (bi-Lipschitz parametrization)
Suppose $E \subset \mathbb{R}^n$ is countably $\mathcal{H}^m$-rectifiable and $\mathcal{H}^m$-measurable. Then for every $\lambda > 1$ there exists a sequence of λ-bi-Lipschitz maps $f_i\colon C_i \to f_i(C_i) \subset E$, with $C_i \subset \mathbb{R}^m$ compact, such that the $f_i(C_i)$ are pairwise disjoint and

$$\mathcal{H}^m\big(E \setminus \textstyle\bigcup_i f_i(C_i)\big) = 0.$$

See [Fed, 3.2.18] and [AmbK2, 4.1].

Proof: First we assume that E is a Borel set contained in the image of a single Lipschitz map $h\colon \mathbb{R}^m \to \mathbb{R}^n$. Using Lemma 1.10 (Borel partition), the area formula 1.12, and the inner regularity of $\mathcal{L}^m$, we find a sequence of λ-bi-Lipschitz maps $g_k\colon D_k \to g_k(D_k) \subset E$, with $D_k \subset \mathbb{R}^m$ compact, such that $\mathcal{H}^m(E \setminus \bigcup_k g_k(D_k)) = 0$. Consider the Borel sets

$$D_k' := D_k \setminus g_k^{-1}\big(\textstyle\bigcup_{j=1}^{k-1} g_j(D_j)\big).$$

Then $\bigcup_k g_k(D_k') = \bigcup_k g_k(D_k)$, and the $g_k(D_k')$ are pairwise disjoint. For every k, choose a sequence of pairwise disjoint compact sets $C_{k,l} \subset D_k'$ such that $\mathcal{L}^m(D_k' \setminus \bigcup_l C_{k,l}) = 0$. It follows that $\mathcal{H}^m(E \setminus \bigcup_{k,l} g_k(C_{k,l})) = 0$, and the $g_k(C_{k,l})$ are pairwise disjoint.

To prove the general result, partition E into a sequence of $\mathcal{H}^m$-measurable sets E_j with $\mathcal{H}^m(E_j) < \infty$ and $E_j \subset h_j(\mathbb{R}^m)$ for some Lipschitz map $h_j\colon \mathbb{R}^m \to \mathbb{R}^n$. Then E_j contains an F_σ set F_j with $\mathcal{H}^m(E_j \setminus F_j) = 0$. Now apply the above result to each F_j. □

Suppose X is a metric space, $A \subset X$, and $x \in X$. Recall that the m-dimensional *upper density* and *lower density* of A at x are defined by

$$\Theta^{*m}(A, x) = \limsup_{r \downarrow 0} \frac{\mathcal{H}^m(A \cap \mathrm{B}(x,r))}{\alpha_m r^m},$$
$$\Theta_*^m(A, x) = \liminf_{r \downarrow 0} \frac{\mathcal{H}^m(A \cap \mathrm{B}(x,r))}{\alpha_m r^m},$$

where $\alpha_m := \mathcal{L}^m(\mathrm{B}^m)$. If the two coincide, then the common value $\Theta^m(A, x)$ is the *density* of A at x.

If $A, B \subset X$ are two $\mathcal{H}^m$-measurable sets with $A \subset B$ and $\mathcal{H}^m(B) < \infty$, then

$$2^{-m} \leq \Theta^{*m}(B, x) \leq 1$$

for $\mathcal{H}^m$-almost all $x \in B$,

$$\Theta^m(B, x) = 0$$

for $\mathcal{H}^m$-almost all $x \in X \setminus B$, and

$$\Theta^{*m}(A, x) = \Theta^{*m}(B, x), \quad \Theta^m_*(A, x) = \Theta^m_*(B, x)$$

for $\mathcal{H}^m$-almost all $x \in A$. (See e.g. [Mat, 6.2, 6.3].)

Also recall Lebesgue's theorem: If $u \in L^1(\mathbb{R}^m)$, then $\mathcal{L}^m$-almost every point x is a *Lebesgue point* of u, i.e.

$$\lim_{r \downarrow 0} \frac{1}{\alpha_m r^m} \int_{\mathrm{B}(x,r)} |u(y) - u(x)| \, dy = 0.$$

For $x \in \mathbb{R}^n$ and $r > 0$, define $T_{x,r} \colon \mathbb{R}^n \to \mathbb{R}^n$, $T_{x,r}(y) = (y - x)/r$. Note that $T_{x,r}$ takes $\mathrm{B}(x, r)$ to $\mathrm{B}(0, 1) = \mathrm{B}^n$.

1.16 Definition (approximate tangent space)
Suppose $E \subset \mathbb{R}^n$ is a $\mathcal{H}^m$-measurable set with $\mathcal{H}^m(E) < \infty$. Let $x \in \mathbb{R}^n$. An m-dimensional linear subspace $L \subset \mathbb{R}^n$ is called the ($\mathcal{H}^m$-)*approximate tangent space* of E at x if

$$\lim_{r \downarrow 0} \int_{T_{x,r}(E)} \varphi \, d\mathcal{H}^m = \int_L \varphi \, d\mathcal{H}^m$$

for all $\varphi \in C_c(\mathbb{R}^n)$. Then we write $L =: \mathrm{Tan}^m(E, x)$.

Clearly $\mathrm{Tan}^m(E, x)$ is uniquely determined if it exists. There are various definitions of approximate tangent spaces in the literature, compare [Sim, 11.2], [Fed, 3.2.16], and [Mat, 15.17].

1.17 Theorem (existence of tangent spaces)
Suppose $E \subset \mathbb{R}^n$ is a countably $\mathcal{H}^m$-rectifiable and $\mathcal{H}^m$-measurable set with $\mathcal{H}^m(E) < \infty$. Then for $\mathcal{H}^m$-almost every $x \in E$, $\mathrm{Tan}^m(E, x)$ exists and $\Theta^m(E, x) = 1$.

For this result and Theorem 1.18 below, see [Fed, 3.2.19], [Sim, 11.6], and [Mat, 15.19].

Proof: Choose a sequence of m-dimensional C^1 submanifolds M_k of $\mathbb{R}^n$ such that $\mathcal{H}^m(E \setminus \bigcup_k M_k) = 0$, cf. 1.14. Put $E_k := E \cap M_k$; then $\mathcal{H}^m(E \setminus \bigcup_k E_k) = 0$. Since M_k is C^1, it follows that for $\mathcal{H}^m$-almost every $x \in E_k$, we have $\Theta^m(E_k, x) = 1$ and $\mathrm{Tan}^m(E_k, x) = T_x M_k$. Moreover, for $\mathcal{H}^m$-almost every $x \in E_k$, $\Theta^m(E \setminus E_k, x) = 0$. Combining these two properties we conclude that for $\mathcal{H}^m$-almost every $x \in E_k$, $\Theta^m(E, x) = 1$ and $\mathrm{Tan}^m(E, x) = T_x M_k$. □

The following two converses to 1.17 hold. The second is a deep result of Preiss [Pre]; an account of the theorem and its history is given in [Mat, Sect. 17].

1.18 Theorem
Suppose $E \subset \mathbb{R}^n$ is a $\mathcal{H}^m$-measurable set with $\mathcal{H}^m(E) < \infty$. If $\mathrm{Tan}^m(E, x)$ exists for $\mathcal{H}^m$-almost every $x \in E$, then E is countably $\mathcal{H}^m$-rectifiable.

1.19 Theorem (Preiss)
Suppose $E \subset \mathbb{R}^n$ is a $\mathcal{H}^m$-measurable set with $\mathcal{H}^m(E) < \infty$. If the density $\Theta^m(E, x)$ exists for $\mathcal{H}^m$-almost every $x \in E$, then E is countably $\mathcal{H}^m$-rectifiable.

Finally, we state the Besicovitch–Federer projection theorem which played a very important role in the development of the theory of currents. This deep result was proved in [Bes] for $m = 1$ and $n = 2$ and in [Fed0] for general dimensions. See [Fed, 3.3.13] and [Mat, 18.1]. A set $F \subset \mathbb{R}^n$ is *purely $\mathcal{H}^m$-unrectifiable* if $\mathcal{H}^m(F \cap f(\mathbb{R}^m)) = 0$ for every Lipschitz map $f \colon \mathbb{R}^m \to \mathbb{R}^n$. Every set $A \subset \mathbb{R}^n$ with $\mathcal{H}^m(A) < \infty$ can be written as the disjoint union of a countably $\mathcal{H}^m$-rectifiable set E and a purely $\mathcal{H}^m$-unrectifiable set F (cf. [Mat, 15.6]).

1.20 Theorem (Besicovitch, Federer)
Suppose $F \subset \mathbb{R}^n$ is a purely $\mathcal{H}^m$-unrectifiable set with $\mathcal{H}^m(F) < \infty$. Then for $\gamma_{n,m}$-almost every $L \in \mathrm{G}(n, m)$, $\mathcal{H}^m(\pi_L(F)) = 0$. Here $\gamma_{n,m}$ denotes the Haar measure on $\mathrm{G}(n, m)$, and $\pi_L \colon \mathbb{R}^n \to L$ is orthogonal projection.

Lecture 2: Normal currents

Vectors, covectors, and forms

Denote by $e_1, \dots, e_n$ the standard basis for $\mathbb{R}^n$ and by $e_1^*, \dots, e_n^*$ the dual basis for the dual space $(\mathbb{R}^n)^* = \{f\colon \mathbb{R}^n \to \mathbb{R} \text{ linear}\}$, such that $e_i^*(e_j) = \delta_{ij}$ for all i, j.

For $m \in \mathbb{N}$, $\Lambda_m\mathbb{R}^n$ and $\Lambda^m\mathbb{R}^n$ denote the vector spaces of *m-vectors* and *m-covectors* of $\mathbb{R}^n$, respectively. In case $1 \le m \le n$, a basis of $\Lambda_m\mathbb{R}^n$ is given by

$$\{e_\lambda := e_{\lambda(1)} \wedge \ldots \wedge e_{\lambda(m)} \colon \lambda \in \Lambda(n,m)\},$$

where $\Lambda(n,m)$ denotes the set of all strictly increasing maps from $\{1,\dots,m\}$ into $\{1,\dots,n\}$. Similarly,

$$\{e_\lambda^* = e_{\lambda(1)}^* \wedge \ldots \wedge e_{\lambda(m)}^* \colon \lambda \in \Lambda(n,m)\}$$

is a basis of $\Lambda^m\mathbb{R}^n$. For $m > n$, $\Lambda_m\mathbb{R}^n = \Lambda^m\mathbb{R}^n = \{0\}$. By convention, $\Lambda_0\mathbb{R}^n = \Lambda^0\mathbb{R}^n = \mathbb{R}$.

An m-vector τ is *simple* if it can be written as a product of m vectors, $\tau = v_1 \wedge \ldots \wedge v_m$. Simple covectors are define analogously.

We write $\langle \tau, \omega \rangle$ for the duality product of $\tau \in \Lambda_m\mathbb{R}^n$ and $\omega \in \Lambda^m\mathbb{R}^n$, thus $\langle e_\lambda, e_\mu^* \rangle = \delta_{\lambda\mu}$ for $\lambda, \mu \in \Lambda(n,m)$.

The standard inner product $\langle \cdot, \cdot \rangle$ and euclidean norm $|\cdot|$ on $\mathbb{R}^n$ induce corresponding inner products and norms on $\Lambda_m\mathbb{R}^n$ and $\Lambda^m\mathbb{R}^n$ such that the above bases are orthonormal. They will be denoted by the same symbols $\langle \cdot, \cdot \rangle$ and $|\cdot|$. For $\tau = v_1 \wedge \ldots \wedge v_m \in \Lambda_m\mathbb{R}^n$,

$$|\tau| = \sqrt{\det(\langle v_i, v_j \rangle)}.$$

The *comass norm* of an m-covector ω is defined by

$$\|\omega\| = \sup\{\langle \tau, \omega \rangle \colon \tau \in \Lambda_m\mathbb{R}^n \text{ is simple and } |\tau| \le 1\}.$$

Always $\|\omega\| \le |\omega|$, with equality iff ω is simple. The *mass norm* of an m-vector τ is defined by

$$\|\tau\| = \sup\{\langle \tau, \omega \rangle \colon \omega \in \Lambda^m\mathbb{R}^n \text{ and } \|\omega\| \le 1\}.$$

Always $\|\tau\| \ge |\tau|$, with equality iff τ is simple.

By a *C^∞ differential m-form ω on $\mathbb{R}^n$* we mean an m-covectorfield $\omega \in C^\infty(\mathbb{R}^n, \Lambda^m\mathbb{R}^n)$. We denote by

$$\mathcal{D}^m(\mathbb{R}^n) := C_c^\infty(\mathbb{R}^n, \Lambda^m\mathbb{R}^n)$$

the vector space of all m-forms on $\mathbb{R}^n$ with compact support. As usual, we write $dx^\lambda = dx^{\lambda(1)} \wedge \ldots \wedge dx^{\lambda(m)}$ for the constant covectorfield mapping x to $e^*_\lambda = e^*_{\lambda(1)} \wedge \ldots \wedge e^*_{\lambda(m)}$; then $\omega \in \mathcal{D}^m(\mathbb{R}^n)$ is of the form

$$\omega = \sum_{\lambda \in \Lambda(n,m)} \omega_\lambda \, dx^\lambda, \quad \omega_\lambda \in C^\infty_c(\mathbb{R}^n).$$

We equip $\mathcal{D}^m(\mathbb{R}^n)$ with the topology in which

$$\omega^i = \sum_\lambda \omega^i_\lambda dx^\lambda \to 0 \quad \text{for } i \to \infty$$

if and only if there exists a compact set $C \subset \mathbb{R}^n$ with $\operatorname{spt} \omega^i \subset C$ for all i and

$$\sup_x \bigl| D^\alpha \omega^i_\lambda(x) \bigr| \to 0 \quad \text{for } i \to \infty$$

whenever $\lambda \in \Lambda(n,m)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_k \in \mathbb{N} \cup \{0\}$,

$$D^\alpha \omega^i_\lambda = \frac{\partial^{\alpha_1} \cdots \partial^{\alpha_n}}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}} \omega^i_\lambda.$$

Currents

General currents were introduced by deRham, cf. [deR]. The 0-dimensional currents are exactly the distributions in the sense of Schwartz [Sch].

2.1 Definition (current)
An *m-dimensional current* or *m-current* in $\mathbb{R}^n$ is a continuous linear functional on $\mathcal{D}^m(\mathbb{R}^n)$. $\mathcal{D}_m(\mathbb{R}^n)$ denotes the vector space of all m-currents in $\mathbb{R}^n$.

A sequence $(T_i)_{i \in \mathbb{N}}$ in $\mathcal{D}_m(\mathbb{R}^n)$ *converges weakly* to a current $T \in \mathcal{D}_m(\mathbb{R}^n)$ if $\lim_{i \to \infty} T_i(\omega) = T(\omega)$ for all $\omega \in \mathcal{D}^m(\mathbb{R}^n)$; we then write

$$T_i \rightharpoonup T.$$

The *support* $\operatorname{spt} T$ of a current $T \in \mathcal{D}_m(\mathbb{R}^n)$ is the smallest closed set $C \subset \mathbb{R}^n$ with the property that $T(\omega) = 0$ for all $\omega \in \mathcal{D}^m(\mathbb{R}^n)$ with $\operatorname{spt} \omega \cap C = \emptyset$.

2.2 Definition (boundary of a current)
Let $T \in \mathcal{D}_m(\mathbb{R}^n)$, $m \geq 1$. The *boundary of* T is the current $\partial T \in \mathcal{D}_{m-1}(\mathbb{R}^n)$ defined by

$$\partial T(\pi) := T(d\pi) \quad \text{for all } \pi \in \mathcal{D}^{m-1}(\mathbb{R}^n).$$

Clearly $\partial \circ \partial = 0$ since $d \circ d = 0$, $\operatorname{spt} \partial T \subset \operatorname{spt} T$, and $T_i \rightharpoonup T$ implies $\partial T_i \rightharpoonup \partial T$.

2.3 Definition (total variation measure and mass)
Let $T \in \mathcal{D}_m(\mathbb{R}^n)$. For $U \subset \mathbb{R}^n$ open and $A \subset \mathbb{R}^n$ arbitrary, put

$$\|T\|(U) := \sup\{T(\omega)\colon \operatorname{spt}\omega \subset U, \sup_x\|\omega(x)\| \le 1\},$$
$$\|T\|(A) := \inf\{\|T\|(U)\colon U \text{ is open}, A \subset U\}.$$

This defines a Borel regular outer measure $\|T\|$ on $\mathbb{R}^n$.

$$\mathbf{M}(T) := \|T\|(\mathbb{R}^n) \in [0,\infty]$$

is the *mass* of T. We denote by $\mathbf{M}_m(\mathbb{R}^n)$ the vector space of all $T \in \mathcal{D}_m(\mathbb{R}^n)$ with $\mathbf{M}(T) < \infty$. A current $T \in \mathcal{D}_m(\mathbb{R}^n)$ has *locally finite mass* if $\|T\|$ is a Radon measure, i.e. if it is finite on compact sets, and $\mathbf{M}_{m,\mathrm{loc}}(\mathbb{R}^n)$ denotes the vector space of all such currents.

Compare [Fed, 4.1.7] and [Sim, 26.6]. For a fixed open set $U \subset \mathbb{R}^n$, the map $T \mapsto \|T\|(U)$ is lower semicontinuous on $\mathcal{D}_m(\mathbb{R}^n)$ with respect to weak convergence, i.e.,

$$\|T\|(U) \le \liminf_{i\to\infty} \|T_i\|(U) \quad \text{for } T_i \rightharpoonup T.$$

The space $\mathbf{M}_m(\mathbb{R}^n)$ endowed with the norm $\mathbf{M}$ is a Banach space. Note that
$$|T(\omega)| \le \sup_x\|\omega(x)\|\,\mathbf{M}(T)$$
for all $\omega \in \mathcal{D}^m(\mathbb{R}^n)$.

2.4 Example
Suppose $M \subset \mathbb{R}^n$ is an oriented m-dimensional C^1-submanifold with boundary (possibly $\partial M = \emptyset$), and M is a closed subset of $\mathbb{R}^n$. We view the orientation of M as a continuous function $\tau\colon M \to \Lambda_m\mathbb{R}^n$ such that for every $x \in M$, $\tau(x)$ is simple and represents the tangent space T_xM, and $|\tau(x)| = 1$. Then
$$[M](\omega) := \int_M \langle \tau(x), \omega(x)\rangle \, d\mathcal{H}^m(x)$$
defines an m-current $[M] = [M,\tau] \in \mathcal{D}_m(\mathbb{R}^n)$. Note that this integral corresponds to the usual $\int_M \omega$ in the notation of differential geometry.

Suppose ∂M is equipped with the induced orientation $\tau'\colon \partial M \to \Lambda_{m-1}\mathbb{R}^n$, i.e. $\tau = \eta \wedge \tau'$ for the exterior unit normal η. Then we have

$$\begin{aligned}\partial[M](\pi) = [M](d\pi) &= \int_M \langle \tau, d\pi\rangle \, d\mathcal{H}^m \\ &= \int_{\partial M} \langle \tau', \pi\rangle \, d\mathcal{H}^{m-1} = [\partial M](\pi)\end{aligned}$$

for all $\pi \in \mathcal{D}^{m-1}(\mathbb{R}^n)$ by the Theorem of Stokes.

The measure $\|[M]\|$ is simply the restriction of $\mathcal{H}^m$ to M, $\|[M]\|(A) = (\mathcal{H}^m \llcorner M)(A) = \mathcal{H}^m(A \cap M)$.

Whenever μ is a Radon measure on $\mathbb{R}^n$ and $\tau\colon \mathbb{R}^n \to \Lambda_m\mathbb{R}^n$ is locally μ-integrable, then we obtain a current $T = [\mu, \tau] \in \mathcal{D}_m(\mathbb{R}^n)$ by defining

$$T(\omega) := \int_{\mathbb{R}^n} \langle \tau(x), \omega(x)\rangle \, d\mu(x).$$

We say that a current $T \in \mathcal{D}_m(\mathbb{R}^n)$ is *representable by integration* if it admits such a representation; then clearly $T \in \mathbf{M}_{m,\mathrm{loc}}(\mathbb{R}^n)$. In fact, $\mathbf{M}_{m,\mathrm{loc}}(\mathbb{R}^n)$ is precisely the set of m-currents representable by integration:

2.5 Theorem (integral representation)
Let $T \in \mathbf{M}_{m,\mathrm{loc}}(\mathbb{R}^n)$. There is a $\|T\|$-measurable function $\vec{T}\colon \mathbb{R}^n \to \Lambda_m\mathbb{R}^n$ such that $\|\vec{T}(x)\| = 1$ for $\|T\|$-almost every $x \in \mathbb{R}^n$ and

$$T(\omega) = \int_{\mathbb{R}^n} \langle \vec{T}(x), \omega(x)\rangle \, d\|T\|(x) \quad \text{for all } \omega \in \mathcal{D}^m(\mathbb{R}^n).$$

In brief, $T = [\|T\|, \vec{T}]$. This follows from an appropriate version of the Riesz respresentation theorem, cf. [Fed, 4.1.5], [Sim, 26.7].

The *restriction* of $T = [\mu, \tau] \in \mathbf{M}_{m,\mathrm{loc}}(\mathbb{R}^n)$ to a function $u \in L^1_{\mathrm{loc}}(\mu)$ is the current $T \llcorner u \in \mathbf{M}_{m,\mathrm{loc}}(\mathbb{R}^n)$ defined by

$$(T \llcorner u)(\omega) := \int_{\mathbb{R}^n} \langle \tau(x), \omega(x)\rangle u(x) \, d\mu(x).$$

If $B \subset \mathbb{R}^n$ is a Borel set and χ_B is the characteristic function, then we write $T \llcorner B$ for $T \llcorner \chi_B$. (A general current $T \in \mathcal{D}_m(\mathbb{R}^n)$ can be restricted to a function $f \in C^\infty(\mathbb{R}^n)$: $(T \llcorner f)(\omega) := T(f\omega)$.)

2.6 Theorem (weak compactness in $\mathbf{M}_{m,\mathrm{loc}}$)
Suppose $(T_i)_{i \in \mathbb{N}}$ is a sequence in $\mathbf{M}_{m,\mathrm{loc}}(\mathbb{R}^n)$ with $\sup_i \|T_i\|(U) < \infty$ for all open sets $U \subset \mathbb{R}^n$ with compact closure. Then there is a subsequence (T_{i_j}) and a $T \in \mathbf{M}_{m,\mathrm{loc}}(\mathbb{R}^n)$ such that $T_{i_j} \rightharpoonup T$.

This is an application of the Banach–Alaoglu theorem.

2.7 Example
Choose $\eta \in C^\infty_c(\mathbb{R})$ with $\eta \geq 0$ and $\int \eta \, dx = 1$, and let $\eta_i(x) := i\eta(ix)$ for $i = 1, 2, \ldots$. Define $T_i \in \mathcal{D}_1(\mathbb{R})$ by $T_i := [\mathbb{R}] \llcorner \eta_i$, i.e.

$$T_i(\omega) = \int \langle 1, \omega(x)\rangle \eta_i(x) \, dx.$$

Then $\mathbf{M}(T_i) = \int \eta_i \, dx = 1$ for all i, and $T_i \rightharpoonup T$ for the current T satisfying

$$T(\omega) = \langle 1, \omega(0)\rangle.$$

Note that

$$\partial T_i(f) = T_i(df) = \int \langle 1, df\rangle \eta_i \, dx = \int f' \eta_i \, dx = -\int f \eta_i' \, dx,$$

$\mathbf{M}(\partial T_i) = \int |\eta_i'| \, dx$, and $\partial T(f) = f'(0)$, $\mathbf{M}(\partial T) = \infty$.

2.8 Definition (push-forward)
The *push-forward* of a current $T \in \mathcal{D}_m(\mathbb{R}^n)$ under a C^∞ map f from $\mathbb{R}^n$ into $\mathbb{R}^p$ is defined as follows. Suppose $f|\operatorname{spt} T$ is proper, i.e. $\operatorname{spt} T \cap f^{-1}(C)$ is compact whenever $C \subset \mathbb{R}^p$ is compact. Given a form $\omega \in \mathcal{D}^m(\mathbb{R}^p)$, consider its pull-back $f^\# \omega$, pick a function $\zeta_\omega \in C_c^\infty(\mathbb{R}^n)$ such that $\zeta_\omega \equiv 1$ in a neighborhood of the compact set $\operatorname{spt} T \cap \operatorname{spt}(f^\# \omega) \subset \operatorname{spt} T \cap f^{-1}(\operatorname{spt} \omega)$, and put

$$(f_\# T)(\omega) := T(\zeta_\omega f^\# \omega).$$

This defines a current $f_\# T \in \mathcal{D}_m(\mathbb{R}^p)$; the definition is independent of the choice of the functions ζ_ω.

The following properties hold:

$$\partial(f_\# T) = f_\#(\partial T), \quad \operatorname{spt}(f_\# T) \subset f(\operatorname{spt} T),$$

and $(g \circ f)_\# T = g_\#(f_\# T)$ whenever g is a C^∞ map from $\mathbb{R}^p$ into $\mathbb{R}^q$ such that $g \circ f|\operatorname{spt} T$ is proper. If $T \in \mathbf{M}_{m,\mathrm{loc}}(\mathbb{R}^n)$, $T = [\|T\|, \vec{T}]$, then

$$f_\# T(\omega) = \int_{\mathbb{R}^n} \big\langle Df_{x\#} \vec{T}(x), \omega(f(x)) \big\rangle \, d\|T\|(x)$$

for all $\omega \in \mathcal{D}^m(\mathbb{R}^p)$, hence

$$\|f_\# T\|(V) \le \int_{f^{-1}(V)} \|Df_{x\#} \vec{T}(x)\| \, d\|T\|(x)$$

for all open sets $V \subset \mathbb{R}^p$. Thus $f_\# T \in \mathbf{M}_{m,\mathrm{loc}}(\mathbb{R}^p)$. See [Fed, p. 359], [Sim, 26.21].

Normal currents

The theory of normal and integral currents was initiated by [FedF].

2.9 Definition (normal current)
Let $T \in \mathcal{D}_m(\mathbb{R}^n)$, $m \ge 1$. Put

$$\mathbf{N}(T) := \mathbf{M}(T) + \mathbf{M}(\partial T).$$

T is called *normal* if $\mathbf{N}(T) < \infty$ and *locally normal* if $\|T\| + \|\partial T\|$ is a Radon measure. The respective vector spaces are denoted $\mathbf{N}_m(\mathbb{R}^n)$ and $\mathbf{N}_{m,\mathrm{loc}}(\mathbb{R}^n)$. For $m = 0$, $\mathbf{N}(T) := \mathbf{M}(T)$, $\mathbf{N}_0(\mathbb{R}^n) := \mathbf{M}_0(\mathbb{R}^n)$ and $\mathbf{N}_{0,\mathrm{loc}}(\mathbb{R}^n) := \mathbf{M}_{0,\mathrm{loc}}(\mathbb{R}^n)$.

Note that the space $\mathbf{N}_m(\mathbb{R}^n)$ endowed with the norm $\mathbf{N}$ is a Banach space, and the compactness theorem 2.6 holds with $\mathbf{M}_{m,\mathrm{loc}}(\mathbb{R}^n)$ and $\|T_i\|(U)$ replaced by $\mathbf{N}_{m,\mathrm{loc}}(\mathbb{R}^n)$ and $(\|T_i\| + \|\partial T_i\|)(U)$. (In [Fed], $T \in \mathbf{N}_m(\mathbb{R}^n)$ means in addition that $\operatorname{spt} T$ is compact.)

For a compactly supported current $T \in \mathbf{N}_m(\mathbb{R}^n)$, the definition of $f_\# T$ given in 2.8 can be extended to the case that $f \colon \mathbb{R}^n \to \mathbb{R}^p$ is a locally Lipschitz map (cf. [Fed, 4.1.14], [Sim, 26.25]). The idea is as follows. Pick a symmetric mollifier $\eta \in C_c^\infty(\mathbb{R}^n)$ (so that $\eta \ge 0$, $\operatorname{spt} \eta \subset \mathrm{B}(0,1)$, $\eta(-x) = \eta(x)$, and $\int \eta \, dx = 1$). For $\epsilon > 0$, put $\eta_\epsilon(x) = \eta(x/\epsilon)/\epsilon^n$ and consider the mollified functions $f^{(\epsilon)} := \eta_\epsilon * f$,

$$(\eta_\epsilon * f)(x) := \int \eta_\epsilon(x-y) f(y) \, dy.$$

Now show that, as $\epsilon \to 0$, $f^{(\epsilon)}_\# T$ converges weakly to a current $S \in \mathbf{N}_m(\mathbb{R}^p)$ that does not depend on the choice of η. Then define $f_\# T := S$. The properties

$$\partial(f_\# T) = f_\#(\partial T), \quad \operatorname{spt}(f_\# T) \subset f(\operatorname{spt} T),$$

and $(g \circ f)_\# T = g_\#(f_\# T)$ then hold for all locally Lipschitz maps $f \colon \mathbb{R}^n \to \mathbb{R}^p$ and $g \colon \mathbb{R}^p \to \mathbb{R}^q$, and

$$\mathbf{M}(f_\# T) \le \operatorname{Lip}(f|\operatorname{spt} T)\mathbf{M}(T).$$

The proof of the existence of $f_\# T$ uses homotopies of currents, which we describe next.

We need some notation for 0- and 1-dimensional currents. For $a \in \mathbb{R}^n$ we define $[a] \in \mathcal{D}_0(\mathbb{R}^n)$ by

$$[a](f) := f(a) \quad \text{for } f \in \mathcal{D}^0(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n);$$

$[a]$ corresponds to δ_a in the notation of distribution theory. For $[a,b] \subset \mathbb{R}$ we denote by $[a,b] \in \mathcal{D}_1(\mathbb{R})$ the current satisfying

$$[a,b](\varphi \, dt) = \int_a^b \varphi(t) \, dt \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}).$$

We have $\partial[a,b](f) = [a,b](df) = [a,b](f' \, dt) = \int_a^b f' \, dt = f(b) - f(a) = [b](f) - [a](f)$ for all $f \in \mathcal{D}^0(\mathbb{R})$, i.e.

$$\partial[a,b] = [b] - [a].$$

Next, for $[a,b] \subset \mathbb{R}$, we define the *cartesian product* of $[a] \in \mathcal{D}_0(\mathbb{R})$ or $[a,b] \in \mathcal{D}_1(\mathbb{R})$ with a current $T \in \mathcal{D}_m(\mathbb{R}^n)$. We put

$$[a] \times T := i_{a\#} T \in \mathcal{D}_m(\mathbb{R} \times \mathbb{R}^n),$$

where $i_a \colon \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$, $i_a(x) = (a,x)$. In case $m \ge 1$,

$$\partial([a] \times T) = \partial(i_{a\#} T) = i_{a\#}(\partial T) = [a] \times \partial T.$$

To define $[a,b] \times T \in \mathcal{D}_{m+1}(\mathbb{R} \times \mathbb{R}^n)$, denoting the canonical coordinates on $\mathbb{R} \times \mathbb{R}^n$ by (t,x), we write $\omega \in \mathcal{D}^{m+1}(\mathbb{R} \times \mathbb{R}^n)$ in the form $dt \wedge \omega' + \bar{\omega}$ such that $dt \wedge \omega'$ collects all terms containing dt; then

$$([a,b] \times T)(\omega) := \int_a^b ([t] \times T)(\omega')\, dt.$$

Cartesian products of two general currents are defined in [Fed, 4.1.8] and [Sim, 26.16].

2.10 Lemma (interval $\times$ current)
Let $T \in \mathcal{D}_m(\mathbb{R}^n)$. Then $\operatorname{spt}([a,b] \times T) = [a,b] \times \operatorname{spt} T$,

$$\partial([a,b] \times T) = [b] \times T - [a] \times T - [a,b] \times \partial T$$

for $m \geq 1$, and $\partial([a,b] \times T) = [b] \times T - [a] \times T$ for $m = 0$. If $T \in \mathbf{M}_{m,\mathrm{loc}}(\mathbb{R}^n)$, then $\|[a,b] \times T\| = \|[a,b]\| \times \|T\|$.

Now suppose $f, g \colon \mathbb{R}^n \to \mathbb{R}^p$ are two C^∞ maps. Let h be a smooth homotopy from f to g, i.e. a C^∞ map from an open neighborhood of $[0,1] \times \mathbb{R}^n$ in $\mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}^p$ with $h(0,x) = f(x)$ and $h(1,x) = g(x)$ for all $x \in \mathbb{R}^n$.

2.11 Theorem (homotopy formula)
Suppose $T \in \mathcal{D}_m(\mathbb{R}^n)$, and $h|([0,1] \times \operatorname{spt} T)$ is proper. Then

$$g_\# T - f_\# T = \partial h_\#([0,1] \times T) + h_\#([0,1] \times \partial T)$$

for $m \geq 1$, and $g_\# T - f_\# T = \partial h_\#([0,1] \times T)$ for $m = 0$. If moreover $T \in \mathbf{M}_m(\mathbb{R}^n)$, $\|Df_x\|, \|Dg_x\| \leq \lambda$ for all $x \in \operatorname{spt} T$, and h is the affine homotopy from f to g, i.e. $h(t,x) = (1-t)f(x) + tg(x)$, then

$$\mathbf{M}(h_\#([0,1] \times T)) \leq \sup_{x \in \operatorname{spt} T} |g(x) - f(x)|\, \lambda^m \mathbf{M}(T).$$

See [Fed, 4.1.9], [Sim, 26.22, 26.23].

Finally, we define the *cone* $[a] \times\!\!\!\!\times T \in \mathcal{D}_{m+1}(\mathbb{R}^n)$ from a point $a \in \mathbb{R}^n$ over a current $T \in \mathcal{D}_m(\mathbb{R}^n)$ with compact support. Let $h(t,x) := (1-t)a + tx$. Then

$$[a] \times\!\!\!\!\times T := h_\#([0,1] \times T) \in \mathcal{D}_{m+1}(\mathbb{R}^n).$$

From 2.11 we get

$$\partial([a] \times\!\!\!\!\times T) = T - h_\#([0,1] \times \partial T) = T - [z] \times\!\!\!\!\times \partial T$$

for $m \geq 1$, in particular, $\partial([a] \times\!\!\!\!\times T) = T$ if $\partial T = 0$.

Suppose $T \in \mathbf{N}_m(\mathbb{R}^n)$, $m \geq 1$. Recall that then the restrictions $T \llcorner B$ and $(\partial T) \llcorner B$ are defined for every Borel set $B \subset \mathbb{R}^n$. If $f \colon \mathbb{R}^n \to \mathbb{R}$ is Lipschitz and $s \in \mathbb{R}$, define the *slices*

$$\begin{aligned}\langle T, f, s+\rangle &:= (\partial T) \llcorner \{f > s\} - \partial(T \llcorner \{f > s\}) \\ &= \partial(T \llcorner \{f \leq s\}) - (\partial T) \llcorner \{f \leq s\}, \\ \langle T, f, s-\rangle &:= \partial(T \llcorner \{f < s\}) - (\partial T) \llcorner \{f < s\} \\ &= (\partial T) \llcorner \{f \geq s\} - \partial(T \llcorner \{f \geq s\}).\end{aligned}$$

For all but countably many s it is true that

$$(\|T\| + \|\partial T\|)(f^{-1}\{s\}) = 0, \quad \text{so} \quad \langle T, f, s+\rangle = \langle T, f, s-\rangle.$$

Note that $\partial\langle T, f, s+\rangle = -\langle \partial T, f, s+\rangle$ if $m \geq 2$.

2.12 Theorem (slices of normal currents)
(1) *For all* $s \in \mathbb{R}$, $\operatorname{spt}\langle T, f, s+\rangle \subset f^{-1}\{s\} \cap \operatorname{spt} T$.
(2) *For all* $s \in \mathbb{R}$,

$$\mathbf{M}(\langle T, f, s+\rangle) \leq \operatorname{Lip}(f) \liminf_{h \to 0+} \|T\|(\{s < f < s+h\})/h.$$

(3) *Whenever* $-\infty \leq a < b \leq \infty$, *then*

$$\int_a^{*b} \mathbf{M}(\langle T, f, s+\rangle)\, ds \leq \operatorname{Lip}(f) \|T\|(\{a < f < b\}).$$

(4) *For almost all* $s \in \mathbb{R}$, $\langle T, f, s+\rangle \in \mathbf{N}_{m-1}(\mathbb{R}^n)$.

See [Fed, 4.2.1], [Sim, 28.6–28.10].

Results for n-currents in $\mathbb{R}^n$

We mention two special results for n-dimensional currents in $\mathbb{R}^n$.

2.13 Theorem (constancy theorem)
Suppose $U \subset \mathbb{R}^n$ *is open and connected,* $T \in \mathcal{D}_n(\mathbb{R}^n)$, *and* $\operatorname{spt} \partial T \subset \mathbb{R}^n \setminus U$. *Then there exists a constant* $c \in \mathbb{R}$ *such that* $\operatorname{spt}(T - c[U]) \cap U = \emptyset$.

Here $[U] := [U, e_1 \wedge \ldots \wedge e_n]$, i.e.

$$[U](\omega) = \int \langle e_1 \wedge \ldots \wedge e_n, \omega\rangle\, dx = \int f\, dx$$

for $\omega = f\, dx^1 \wedge \ldots \wedge dx^n \in \mathcal{D}^n(\mathbb{R}^n)$. See [Fed, p. 357], [Sim, 26.27].

2.14 Theorem (characterizing $\mathbf{N}_{n,\mathrm{loc}}(\mathbb{R}^n)$)
Suppose $T \in \mathcal{D}_n(\mathbb{R}^n)$. *Then* $T \in \mathbf{N}_{n,\mathrm{loc}}(\mathbb{R}^n)$ *if and only if* $T = [\mathbb{R}^n] \llcorner u$ *for some* $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)$, *in which case* $\|\partial T\| = |Du|$.

Cf. [Sim, 26.28], [GiaMS, p. 454].

Lecture 3: Integral currents

Integer rectifiable currents

The following definition generalizes Example 2.4 and relies on Theorem 1.17 (existence of tangent spaces).

3.1 Definition (integer rectifiable current)
A current $T \in \mathcal{D}_m(\mathbb{R}^n)$ is called *locally integer rectifiable* if it admits a representation of the form

$$T(\omega) = \int_E \langle \tau(x), \omega(x) \rangle \theta(x) \, d\mathcal{H}^m(x) \quad \text{where}$$

(1) $E \subset \mathbb{R}^n$ is a countably $\mathcal{H}^m$-rectifiable and $\mathcal{H}^m$-measurable set,
(2) θ is a locally $\mathcal{H}^m$-integrable positive integer-valued function on E,
(3) τ is an $\mathcal{H}^m$-measurable $\Lambda_m \mathbb{R}^n$-valued function on E such that for $\mathcal{H}^m$-almost every $x \in E$, $\tau(x)$ is simple, $|\tau(x)| = 1$, and $\tau(x)$ represents the approximate tangent space $\mathrm{Tan}^m(E, x) \in \mathrm{G}(n, m)$.

We then write $T = [E, \tau, \theta]$, or just $T = [E, \tau]$ in case $\theta \equiv 1$ $\mathcal{H}^m$-almost everywhere on E. The set of locally integer rectifiable currents in $\mathbb{R}^n$ is denoted by $\mathcal{I}_{m,\mathrm{loc}}(\mathbb{R}^n)$. An *integer rectifiable current* in $\mathbb{R}^n$ is an element of $\mathcal{I}_m(\mathbb{R}^n) := \mathcal{I}_{m,\mathrm{loc}}(\mathbb{R}^n) \cap \mathbf{M}_m(\mathbb{R}^n)$.

Here we follow the presentation in [Sim, Sect. 27]. (In [Fed], the space $\mathcal{I}_{m,\mathrm{loc}}(\mathbb{R}^n)$ is denoted by $\mathcal{R}_m^{\mathrm{loc}}(\mathbb{R}^n)$.) If $T, T' \in \mathcal{I}_{m,\mathrm{loc}}(\mathbb{R}^n)$, then $T + T' \in \mathcal{I}_{m,\mathrm{loc}}(\mathbb{R}^n)$. The mass of $T = [E, \tau, \theta] \in \mathcal{I}_{m,\mathrm{loc}}(\mathbb{R}^n)$ in an open set U is given by

$$\|T\|(U) = \int_{U \cap E} \theta \, d\mathcal{H}^m.$$

Note that this is finite if the closure of U is compact, thus $\mathcal{I}_{m,\mathrm{loc}}(\mathbb{R}^n) \subset \mathbf{M}_{m,\mathrm{loc}}(\mathbb{R}^n)$.

If $T \in \mathcal{I}_{m,\mathrm{loc}}(\mathbb{R}^n)$ and $f \colon \mathbb{R}^n \to \mathbb{R}^p$ is a locally Lipschitz map such that $f|\operatorname{spt} T$ is proper, then $f_\# T$ can be defined, and $f_\# T \in \mathcal{I}_{m,\mathrm{loc}}(\mathbb{R}^p)$. Cf. [Sim, 27.2], [Fed, 4.1.30].

The boundary of an integer rectifiable current need not be integer rectifiable.

3.2 Definition (integral current)
The space of *locally integral currents* in $\mathbb{R}^n$ is defined by

$$\mathbf{I}_{m,\mathrm{loc}}(\mathbb{R}^n) := \{T \in \mathcal{I}_{m,\mathrm{loc}}(\mathbb{R}^n) \colon \partial T \in \mathcal{I}_{m-1,\mathrm{loc}}(\mathbb{R}^n)\}$$

if $m \geq 1$, and $\mathbf{I}_{0,\mathrm{loc}}(\mathbb{R}^n) := \mathcal{I}_{0,\mathrm{loc}}(\mathbb{R}^n)$. An *integral current* in $\mathbb{R}^n$ is an element of $\mathbf{I}_m(\mathbb{R}^n) := \mathbf{I}_{m,\mathrm{loc}}(\mathbb{R}^n) \cap \mathbf{N}_m(\mathbb{R}^n)$.

The following result supplements the slicing theorem 2.12, cf. [Sim, 28.1–28.5], [Fed, 4.3.6].

3.3 Theorem (slices of integral currents)
If $T \in \mathbf{I}_m(\mathbb{R}^n)$, $m \geq 1$, and $f\colon \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, then $\langle T, f, s+\rangle \in \mathbf{I}_{m-1}(\mathbb{R}^n)$ for almost all $s \in \mathbb{R}$.

The compactness theorem

We list three fundamental results in the theory of integer rectifiable currents, cf. [Fed, 4.2.15, 4.2.16], [Sim, 32.2].

3.4 Theorem (boundary rectifiability)
If $T \in \mathcal{I}_m(\mathbb{R}^n)$ and $\mathbf{M}(\partial T) < \infty$, then $\partial T \in \mathcal{I}_{m-1}(\mathbb{R}^n)$, i.e. $T \in \mathbf{I}_m(\mathbb{R}^n)$.

3.5 Theorem (rectifiable slices implies rectifiable)
Suppose $T \in \mathbf{M}_m(\mathbb{R}^n)$ and $\partial T = 0$. Suppose further that for every $z \in \mathbb{R}^n$, $\partial(T \llcorner B(z,r)) \in \mathcal{I}_{m-1}(\mathbb{R}^n)$ for almost all r. Then $T \in \mathcal{I}_m(\mathbb{R}^n)$.

3.6 Theorem (closure theorem)
Suppose $T_1, T_2, \ldots \in \mathbf{I}_m(\mathbb{R}^n)$, $\sup_i \mathbf{N}(T_i) < \infty$, and $\bigcup_i \operatorname{spt} T_i \subset C$ for some compact set $C \subset \mathbb{R}^n$. If $T_i \rightharpoonup T$ for some $T \in \mathcal{D}_m(\mathbb{R}^n)$, then $T \in \mathbf{I}_m(\mathbb{R}^n)$.

The original proof of 3.5 relied on the difficult structure theorem 1.20. In [Whi], White gave a simultaneous inductive proof of the above three theorems that does not use structure theory.

By combining 3.6 with the compactness theorem for normal currents one obtains the celebrated result for integral currents:

3.7 Theorem (compactness theorem)
Suppose $T_1, T_2, \ldots \in \mathbf{I}_m(\mathbb{R}^n)$, $\sup_i \mathbf{N}(T_i) < \infty$, and $\bigcup_i \operatorname{spt} T_i \subset C$ for some compact set $C \subset \mathbb{R}^n$. Then there is a subsequence $T_{i_1}, T_{i_2}, \ldots$ and a $T \in \mathbf{I}_m(\mathbb{R}^n)$ such that $T_{i_j} \rightharpoonup T$.

See [Fed, 4.2.17], [Sim, 27.3].

Minimizing currents

From Theorem 3.7 one easily obtains various existence results for area minimizing currents. In general, a current $T \in \mathcal{I}_{m,\mathrm{loc}}(\mathbb{R}^n)$ is called (*absolutely area*) *minimizing* if

$$\mathbf{M}(S) \leq \mathbf{M}(S')$$

whenever $S, S' \in \mathcal{I}_m(\mathbb{R}^n)$, $\|T\| = \|S\| + \|T - S\|$, and $\partial S' = \partial S$. The assumption $\|T\| = \|S\| + \|T - S\|$ means that S is a *piece* of T.

3.8 Theorem (Plateau problem)
Suppose $R \in \mathbf{I}_{m-1}(\mathbb{R}^n)$, $\operatorname{spt} S$ is compact, and $\partial S = 0$. Then there is current $T \in \mathbf{I}_m(\mathbb{R}^n)$ with $\partial T = R$ such that $\mathbf{M}(T) \le \mathbf{M}(T')$ for all $T' \in \mathbf{I}_m(\mathbb{R}^n)$ with $\partial T' = R$.

Of course T is minimizing in the sense defined above.

Proof: Let $\mathcal{T} := \{T' \in \mathbf{I}_m(\mathbb{R}^n) \colon \partial T' = R\}$; $\mathcal{T}$ is non-empty since it contains e.g. the cone $[0] \times T$. Choose a sequence $T_1, T_2, \dots$ in $\mathcal{T}$ such that

$$\lim_{i \to \infty} \mathbf{M}(T_i) = b := \inf\{\mathbf{M}(T') \colon T' \in \mathcal{T}\}.$$

Let $C \subset \mathbb{R}^n$ be a compact ball that contains $\operatorname{spt} R$, and let $\pi \colon \mathbb{R}^n \to C$ denote the nearest point retraction; π is 1-Lipschitz. Since $\mathbf{M}(\pi_\# T_i) \le \mathbf{M}(T_i)$ and $\operatorname{spt}(\pi_\# T_i) \subset C$ for all i, we assume w.l.o.g. that $\operatorname{spt} T_i \subset C$ for all i. Since $\sup_i \mathbf{N}(T_i) < \infty$, by Theorem 3.7 there exist a subsequence $T_{i_1}, T_{i_2}, \dots$ and a $T \in \mathbf{I}_m(\mathbb{R}^n)$ such that $T_{i_j} \rightharpoonup T$. Then also $R = \partial T_{i_j} \rightharpoonup \partial T$, thus $\partial T = R$, and

$$\mathbf{M}(T) \le \liminf_{j \to \infty} \mathbf{M}(T_{i_j}) = b.$$

This proves the result. □

Let $M \subset \mathbb{R}^{n+k}$ be a compact C^1 submanifold of dimension n. For $m \ge 1$, define the abelian groups

$$\mathcal{Z}_m(M) := \{T \in \mathbf{I}_m(\mathbb{R}^{n+k}) \colon \operatorname{spt} T \subset M,\ \partial T = 0\},$$
$$\mathcal{B}_m(M) := \{\partial S \colon S \in \mathbf{I}_{m+1}(\mathbb{R}^{n+k}),\ \operatorname{spt} S \subset M\} \subset \mathcal{Z}_m(M).$$

Two cycles $T, T' \in \mathcal{Z}_m(M)$ are *homologous* if $T - T' \in \mathcal{B}_m(M)$.

3.9 Theorem (homologically minimizing cycles)
Suppose $M \subset \mathbb{R}^{n+k}$ is as above and $T' \in \mathcal{Z}_m(M)$. Among all cycles in $\mathcal{Z}_m(M)$ homologous to T' there is one with minimal mass.

See [Fed, 5.1.6], [Sim, 34.3].

Next, we mention the most important regularity results for minimizing currents in $\mathbb{R}^n$.

3.10 Theorem (hypersurface interior regularity)
Suppose $T \in \mathcal{I}_{n-1,\mathrm{loc}}(\mathbb{R}^n)$ is minimizing. There is a set Σ of Hausdorff dimension at most $n-8$ such that $(\operatorname{spt} T \setminus \operatorname{spt}(\partial T)) \setminus \Sigma$ is a C^∞ submanifold of $\mathbb{R}^n$.

This is due to Federer [Fed1]. It was known previously that the bound $n - 8$ is optimal, by the following example of Bombieri–De Giorgi–Giusti [BomDG].

3.11 Example (minimizing cone)
Let $V := \{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 \colon |x|^2 < |y|^2\}$. Then $\partial[V] \in \mathbf{I}_{7,\mathrm{loc}}(\mathbb{R}^8)$ is a minimizing hypersurface with an isolated singular point at 0.

For currents of arbitrary codimension, the following deep result of Almgren holds, cf. [Alm2], [Alm3].

3.12 Theorem (interior regularity)
Suppose $T \in \mathcal{I}_{m,\mathrm{loc}}(\mathbb{R}^n)$ is minimizing. There is a set Σ of Hausdorff dimension at most $m-2$ such that $(\operatorname{spt} T \setminus \operatorname{spt}(\partial T)) \setminus \Sigma$ is an m-dimensional C^∞ submanifold of $\mathbb{R}^n$.

The bound $m-2$ is optimal;

$$T = [E_{12}, e_1 \wedge e_2] + [E_{34}, e_3 \wedge e_4] \in \mathcal{I}_{2,\mathrm{loc}}(\mathbb{R}^4)$$

is a minimizing 2-current in $\mathbb{R}^4$ with an isolated singular point at 0. Here E_{ij} denotes the coordinate plane spanned by e_i and e_j. To show that T is minimizing, let $S \in \mathcal{I}_2(\mathbb{R}^4)$ be a piece of T, and let $S' \in \mathcal{I}_2(\mathbb{R}^4)$ be an arbitrary current with $\partial S' = \partial S$. Then $S = [A, e_1 \wedge e_2] + [A', e_3 \wedge e_4]$ for some $\mathcal{H}^2$-measurable sets $A \subset E_{12}$ and $A' \subset E_{34}$. By (a special case of) Wirtinger's inequality [Fed, p. 40], the form $\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$ has comass norm $\|\omega(x)\| = 1$. Since ω is exact and $\partial S = \partial S'$, one concludes that

$$\mathbf{M}(S) = S(\omega) = S'(\omega) \le \mathbf{M}(S'),$$

proving that T is minimizing.

A most useful tool in regularity theory is the following property of minimizing currents, cf. [Fed, 5.4.3–5.4.5], [Sim, 35.1].

3.13 Theorem (monotonicity formula)
Suppose $T \in \mathcal{I}_{m,\mathrm{loc}}(\mathbb{R}^n)$ is minimizing.
(1) *For all $x \in \mathbb{R}^n$, the function $r \mapsto \mathbf{M}(T \llcorner \mathrm{B}(x,r))/r^m$ is (non-strictly) increasing on $(0, d(x, \operatorname{spt} \partial T))$.*
(2) *For all $x \in \operatorname{spt} T$ and $r \in (0, d(x, \operatorname{spt} \partial T))$,*

$$\mathbf{M}(T \llcorner \mathrm{B}(x,r)) \ge \alpha_m r^m = \mathcal{L}^m(\mathrm{B}^m(r)).$$

Proof: (1) For $r \in (0, d(x, \operatorname{spt} \partial T))$, put $S_r := T \llcorner \mathrm{B}(x,r)$ and $f(r) := \mathbf{M}(S_r)$. For almost every r, f is differentiable at r, and

$$\mathbf{M}(\partial S_r) = \mathbf{M}(\langle T, d(x,\cdot), r+\rangle) \le f'(r)$$

by Theorem 2.12(2). The cone $S'_r := [x] \divideontimes \partial S_r$ from x over ∂S_r has mass $\mathbf{M}(S'_r) = \frac{r}{m}\mathbf{M}(\partial S_r)$. Thus, by minimality,

$$f(r) = \mathbf{M}(S_r) \le \mathbf{M}(S'_r) \le \frac{r}{m} f'(r)$$

for almost every $r \in (0, d(x, \operatorname{spt} \partial T))$. Integrating the inequality

$$\frac{f'(r)}{f(r)} \geq \frac{m}{r}$$

from r_1 to r_2 one obtains the result.

(2) is a consequence of (1) and the fact that if $T = [E, \tau, \theta]$, where $\theta > 0$, then $\operatorname{spt} T$ is the closure of the set $\{y \in \mathbb{R}^n \colon \Theta^m(E, y) = 1\}$. $\square$

Lecture 4: Some metric space techniques

In this last lecture, we discuss some metric space techniques that are useful in connection with the new metric approach to currents developed in [AmbK1], [AmbK2], and [Wen]. Indispensable tools in modern geometry, they are also of independent interest.

Embeddings

We start with some basic and well-known isometric embedding theorems for metric spaces.

4.1 Lemma (Kuratowski, Fréchet)
(1) *Every metric space X admits an isometric embedding into* $l^\infty(X) = (\{s\colon X \to \mathbb{R}\colon s \text{ bounded}\}, \|\cdot\|_\infty)$.
(2) *Every separable metric space admits an isometric embedding into* $l^\infty = l^\infty(\mathbb{N})$.

Proof: (1) Fix $z \in X$ and define $X \to l^\infty(X)$,

$$x \mapsto s^x, \quad s^x(y) = d(x,y) - d(y,z).$$

Note that $\|s^x\|_\infty = \sup_y |s^x(y)| \le d(x,z)$. Moreover,

$$\|s^x - s^{x'}\|_\infty = \sup_y |d(x,y) - d(x',y)| \le d(x,x'),$$

and equality occurs for $y = x'$.

(2) Embed a dense countable subset of X into l^∞ by means of (1). This extends to an isometric embedding of X. □

Note that if X is bounded, we do not need to subtract the term $d(y,z)$ in the definition of s^x. In this case, the embedding is canonical.

Recall that a metric space X is said to be *precompact* or *totally bounded* if for every $\epsilon > 0$, X can be covered by a finite number of closed balls of radius ϵ. We call a set $Y \subset X$ *ϵ-separated* if $d(y,y') \ge \epsilon$ whenever $y, y' \in Y$, $y \ne y'$. Note that X is precompact if and only if for every $\epsilon > 0$, all ϵ-separated subsets of X are finite. A metric space is compact if and only if it is precompact and complete.

4.2 Definition (uniformly precompact family)
A family $(X_j)_{j\in J}$ of metric spaces is called *uniformly precompact* if for all $\epsilon > 0$ there exists a number $n = n(\epsilon) \in \mathbb{N}$ such that each X_j can be covered by n closed balls of radius ϵ. The family $(X_j)_{j\in J}$ is *uniformly bounded* if $\sup_{j\in J} \operatorname{diam} X_j < \infty$.

4.3 Theorem (Gromov embedding)
Suppose that $(X_j)_{j\in J}$ is a uniformly precompact and uniformly bounded family of metric spaces. Then there is a compact metric space Z such that each X_j admits an isometric embedding into Z.

We follow essentially the original proof from [Gro1].

Proof: For $i \in \mathbb{N}$, let $\epsilon_i := 2^{-i}$ and pick $n_i \in \mathbb{N}$ such that each X_j can be covered by n_i closed balls of radius ϵ_i. Choose a partition of $\mathbb{N}$ into sets N_i, $i \in \mathbb{N}$, with cardinality $\#N_i = n_1 n_2 \dots n_i$, and define a map $\pi\colon \mathbb{N} \setminus N_1 \to \mathbb{N}$ such that for each $i \in \mathbb{N}$,

$$\pi^{-1}(N_i) = N_{i+1} \quad \text{and} \quad \#\pi^{-1}\{k\} = n_{i+1} \text{ for all } k \in N_i.$$

In each X_j, we construct a sequence $(x^j_k)_{k\in\mathbb{N}}$ according to the following inductive scheme. For $i = 1$, the points x^j_k with $k \in N_i = N_1$ are chosen such that the n_1 balls $\mathrm{B}(x^j_k, \epsilon_1)$ cover X_j. For $i \geq 1$, if the $n_1 \dots n_i$ centers x^j_k with $k \in N_i$ are selected, the $n_1 \dots n_i n_{i+1}$ points x^j_l with $l \in N_{i+1}$ are chosen such that for each $k \in N_i$, the ball $\mathrm{B}(x^j_k, \epsilon_i)$ is covered by the n_{i+1} balls

$$\mathrm{B}(x^j_l, \epsilon_{i+1}) \subset \mathrm{B}(x^j_k, 2\epsilon_i)$$

with $l \in \pi^{-1}\{k\}$. This way we obtain for every $j \in J$ a dense sequence $(x^j_k)_{k\in\mathbb{N}}$ in X_j which gives rise to an isometric embedding $f_j\colon X_j \to l^\infty$, mapping x to $(d(x, x^j_k))_{k\in\mathbb{N}}$. Whenever $i \in \mathbb{N}$, $k \in N_i$, and $l \in \pi^{-1}\{k\}$, then

$$|d(x, x^j_k) - d(x, x^j_l)| \leq d(x^j_k, x^j_l) \leq 2\epsilon_i.$$

Hence, each $f_j(X_j)$ lies in the set Z of all sequences $(s_k)_{k\in\mathbb{N}}$ with $0 \leq s_k \leq \sup_j \operatorname{diam} Z_j$ for all $k \in \mathbb{N}$ and

$$|s_k - s_l| \leq 2\epsilon_i \quad \text{whenever } i \in \mathbb{N},\ k \in N_i, \text{ and } l \in \pi^{-1}\{k\}.$$

Since the sequence $(\epsilon_i)_{i\in\mathbb{N}}$ is summable, Z is a compact subset of l^∞. □

For further reading on geometric embedding theorems and detailed references we refer to [Hei].

Gromov–Hausdorff convergence

For subsets A, B of a metric space X we denote by $\mathrm{N}_\delta(A)$ the closed δ-neighborhood of A and by

$$d_H(A, B) = \inf\{\delta \geq 0\colon A \subset \mathrm{N}_\delta(B),\ B \subset \mathrm{N}_\delta(A)\}$$

the *Hausdorff distance* of A and B; d_H defines a metric on the set $\mathcal{C}$ of non-empty, closed and bounded subsets of X.

4.4 Theorem (Blaschke)
Suppose that $X = (X, d)$ is a metric space and $\mathcal{C}$ is the set of non-empty, closed and bounded subsets of X, endowed with the Hausdorff metric d_H.
(1) *If X is complete, then $\mathcal{C}$ is complete.*
(2) *If X is compact, then $\mathcal{C}$ is compact.*

This was first proved by Blaschke [Bla] for compact convex bodies in $\mathbb{R}^3$ to settle the existence question in the isoperimetric problem.

Proof: (1): Let $(C_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{C}$. Then the set

$$C := \bigcap_{i=1}^{\infty} \overline{\bigcup_{j\geq i} C_j}$$

is closed and bounded. We show that

$$\lim_{i\to\infty} d_H(C_i, C) = 0.$$

Let $\epsilon > 0$. Choose i_0 such that $d_H(C_i, C_j) \leq \epsilon/2$ whenever $i, j \geq i_0$. Suppose $x \in C$. Since $C \subset \overline{\bigcup_{j\geq i_0} C_j}$ there exists an index $j \geq i_0$ with $d(x, C_j) \leq \epsilon/2$. Hence $d(x, C_i) \leq d(x, C_j) + d_H(C_i, C_j) \leq \epsilon$ for all $i \geq i_0$. This shows that $C \subset \mathrm{N}_\epsilon(C_i)$ for $i \geq i_0$.

Now suppose $x \in C_i$ for some $i \geq i_0$. Pick a sequence $i = i_1 < i_2 < \dots$ such that $d_H(C_m, C_n) \leq \epsilon/2^k$ whenever $m, n \geq i_k$, $k \in \mathbb{N}$. Then choose a sequence $(x_k)_{k\in\mathbb{N}}$ such that $x_1 = x$, $x_k \in C_{i_k}$ and $d(x_k, x_{k+1}) \leq \epsilon/2^k$. As X is complete, the Cauchy sequence (x_k) converges to some point y. We have

$$d(x, y) = \lim_{k\to\infty} d(x, x_k) \leq \sum_{k=1}^{\infty} d(x_k, x_{k+1}) \leq \epsilon,$$

and y belongs to the closure of $C_{i_k} \cup C_{i_{k+1}} \cup \dots$ for all k. Thus $y \in C$ and $d(x, C) \leq \epsilon$. This shows that $C_i \subset \mathrm{N}_\epsilon(C)$ whenever $i \geq i_0$.

(2): We know that $\mathcal{C}$ is complete since X is, so it suffices to show that $\mathcal{C}$ is precompact. Let $\epsilon > 0$. Since X is precompact we find a finite set $Z \subset X$ with $\mathrm{N}_\epsilon(Z) = X$. We show that every $C \in \mathcal{C}$ is at Hausdorff distance at most ϵ of some subset of Z, namely $Z_C := Z \cap \mathrm{N}_\epsilon(C)$. For every $x \in C$ there exists a point $z \in Z$ with $d(x, z) \leq \epsilon$, so $z \in Z_C$. This shows that $C \subset \mathrm{N}_\epsilon(Z_C)$. Since also $Z_C \subset \mathrm{N}_\epsilon(C)$, we have $d_H(C, Z_C) \leq \epsilon$. As there are only finitely many distinct subsets of Z, we conclude that $\mathcal{C}$ is precompact. $\square$

4.5 Definition (Gromov–Hausdorff distance)
The *Gromov–Hausdorff distance* of two metric spaces X, Y is the number

$$d_{GH}(X, Y) = \inf d_H^Z(X', Y'),$$

where the infimum is taken over all triples (Z, X', Y') such that $Z = (Z, d^Z)$ is a metric space, $X' \subset Z$ is an isometric copy of X, and $Y' \subset Z$ is an isometric copy of Y.

Cf. [Gro1], [Gro2]. Alternatively, call a metric $\bar{d}$ on the disjoint union $X \sqcup Y$ *admissible* for the given metrics $d = d^X$ and $d = d^Y$ on X and Y if $\bar{d}|X \times X = d^X$ and $\bar{d}|Y \times Y = d^Y$; then

$$d_{GH}(X,Y) = \inf \bar{d}_H(X,Y)$$

where the infimum is taken over all admissible metrics $\bar{d}$ on $X \sqcup Y$.

For instance, suppose that $\mathrm{diam}(X), \mathrm{diam}(Y) \leq D < \infty$. Setting $\bar{d}(x,y) = D/2$ for $x \in X$ and $y \in Y$ we obtain an admissible metric on $X \sqcup Y$, in particular $d_{GH}(X,Y) \leq D/2$.

We call a subset X' of a metric space X *ϵ-dense in* X if $d_H(X,X') \leq \epsilon$, i.e. $\mathrm{N}_\epsilon(X') = X$. By a *correspondence* $\sim$ between two sets X, Y we mean a symmetric relation between points in X and points in Y such that every element of X is related to at least one element of Y and vice-versa.

4.6 Lemma

Suppose X, Y are two metric spaces and $\epsilon > 0$.

(1) *If $d_{GH}(X,Y) < \epsilon$, then there is a correspondence $\sim$ between X and Y such that*

$$|d(x,x') - d(y,y')| < 2\epsilon$$

whenever $x, x' \in X$, $y, y' \in Y$, and $x \sim y$, $x' \sim y'$.

(2) *Suppose $X' \subset X$ is ϵ-dense in X, $Y' \subset Y$ is ϵ-dense in Y, and there is a correspondence $\sim$ between X' and Y' such that*

$$|d(x,x') - d(y,y')| \leq \epsilon$$

whenever $x, x' \in X'$, $y, y' \in Y'$, and $x \sim y$, $x' \sim y'$. Then $d_{GH}(X,Y) < 2\epsilon$.

Proof: (1): Let $\bar{d}$ be an admissible metric on $X \sqcup Y$ such that $\bar{d}_H(X,Y) < \epsilon$. Then for every $x \in X$ there is a $y \in Y$ with $\bar{d}(x,y) < \epsilon$, and vice-versa. Define a correspondence in this manner.

(2): Define an admissible metric $\bar{d}$ on $X \sqcup Y$ such that $\bar{d}(x,y) = \inf d(x,x') + d(y,y') + \epsilon/2$ for all $x \in X$ and $y \in Y$, where the infimum is taken over all pairs of points $x' \in X'$ and $y' \in Y'$ with $x' \sim y'$. □

4.7 Theorem

(1) *d_{GH} satisfies the triangle inequality, i.e. $d_{GH}(X,Z) \leq d_{GH}(X,Y) + d_{GH}(Y,Z)$ for all metric spaces X, Y, Z.*

(2) *d_{GH} defines a metric on the set of isometry classes of compact metric spaces.*

See [BurBI, 7.3.16, 7.3.30]. Assertion (2) is no longer true if 'compact' is replaced by 'complete and bounded'.

4.8 Theorem (Gromov compactness criterion)
Suppose that $(X_i)_{i\in\mathbb{N}}$ is a uniformly precompact and uniformly bounded sequence of metric spaces. Then there exist a subsequence $(X_{i_j})_{j\in\mathbb{N}}$ and a compact metric space Z such that (X_{i_j}) Gromov–Hausdorff converges to Z, i.e. $\lim_{j\to\infty} d_{GH}(X_{i_j}, Z) = 0$.

This was proved in [Gro1].

Proof: Combine Theorems 4.3 (Gromov embedding) and 4.4(b) (Blaschke). □

Ultralimits

By a *filter* ϕ on a set I we mean a monotonous set function $\phi\colon 2^I \to \{0,1\}$ with $\phi(\emptyset) = 0$, $\phi(I) = 1$, and $\phi(A \cap B) = 1$ whenever $\phi(A) = \phi(B) = 1$. An *ultrafilter* is a filter that is maximal with respect to the partial order $\leq$. Equivalently, a function $\phi\colon 2^I \to \{0,1\}$ is an ultrafilter iff $\phi(I) = 1$ and ϕ is finitely additive, i.e.,

$$\phi(A) + \phi(B) = \phi(A \cup B) \quad \text{whenever } A \cap B = \emptyset.$$

An ultrafilter ϕ is called *free* or *non-principal* if it is zero on finite sets; in the opposite case, ϕ has a unique atom and is said to be *fixed* or *principal*. To prove the existence of a free ultrafilter ϕ on an infinite set I, start with the *Fréchet filter* which is one exactly on complements of finite sets and apply Zorn's lemma within the class of filters that vanish on finite sets.

In the following ϕ denotes a free ultrafilter on $\mathbb{N}$. Given a sequence $(x_i)_{i\in\mathbb{N}}$ in a topological space X, we call a point $x \in X$ a *ϕ-limit* of (x_i), and we write $\lim_\phi x_i = x$, if

$$\phi(\{i \in \mathbb{N}\colon x_i \in U\}) = 1$$

for every neighborhood U of x.

In case X is a metric space, $\lim_\phi x_i = x$ is equivalent to

$$\phi(\{i \in \mathbb{N}\colon d(x, x_i) < \epsilon\}) = 1 \quad \text{for all } \epsilon > 0$$

and hence to $\lim_\phi d(x, x_i) = 0$.

If X is a compact topological Hausdorff space and $(x_i)_{i\in\mathbb{N}}$ is a sequence in X, then there exists a uniquely determined point $x \in X$ such that $\lim_\phi x_i = x$. Since ϕ is free, every neighborhood of x contains infinitely many elements of (x_i). In particular, for every bounded sequence $(a_i)_{i\in\mathbb{N}}$ of real numbers there is a unique point a such that $\lim_\phi a_i = a$, and a is the limit of some subsequence. One has

$$\lim_\phi a_i + \lim_\phi b_i = \lim_\phi (a_i + b_i), \quad \lambda \lim_\phi a_i = \lim_\phi (\lambda a_i)$$

for all bounded sequences $(a_i), (b_i)$ and all $\lambda \in \mathbb{R}$. If $a_i \le b_i$ for all i, then $\lim_\phi a_i \le \lim_\phi b_i$. It follows that if (x_i) and (y_i) are two bounded sequences in a metric space X, with $\lim_\phi x_i = x$ and $\lim_\phi y_i = y$, then

$$\lim_\phi d(x_i, y_i) = d(x, y)$$

since $|d(x,y) - \lim_\phi d(x_i, y_i)| \le \lim_\phi d(x, x_i) + \lim_\phi d(y, y_i) = 0$.

4.9 Definition (ultralimit)
Suppose $(X_i)_{i\in\mathbb{N}}$ is a sequence of pointed metric spaces $X_i = (X_i, d, *_i)$ and ϕ is a free ultrafilter on $\mathbb{N}$. Denote by $(X_i)_\infty$ the set of all sequences (x_i) with $x_i \in X_i$ and $\sup_i d(x_i, *_i) < \infty$. For $(x_i), (y_i) \in (X_i)_\infty$ the sequence $(d(x_i, y_i))$ is bounded, and $d((x_i), (y_i)) := \lim_\phi d_i(x_i, y_i)$ defines a pseudometric on $(X_i)_\infty$. The *ultralimit* (or *ultraproduct*) $(X_i)_\phi$ of the sequence (X_i) is the set of equivalence classes $(x_i)_\phi$ of elements $(x_i) \in (X_i)_\infty$, where

$$(x_i) \sim (y_i) \quad \text{if and only if} \quad d((x_i), (y_i)) = 0,$$

endowed with the metric defined by $d((x_i)_\phi, (y_i)_\phi) := d((x_i), (y_i))$.

4.10 Lemma
The ultralimit $\bar{X} = (X_i)_\phi$ of a sequence (X_i) of pointed metric spaces is complete.

Proof: Pick a Cauchy sequence $(\bar{x}^\gamma)_{\gamma\in\mathbb{N}}$ in $\bar{X}$ and represent each $\bar{x}^\gamma$ by an element $(x_i^\gamma)_{i\in\mathbb{N}} \in (X_i)_\infty$. By induction, choose sets $\mathbb{N} = N_1 \supset N_2 \supset \dots$ with $\phi(N_\gamma) = 1$ for all γ such that

$$|d(\bar{x}^\alpha, \bar{x}^\beta) - d(x_i^\alpha, x_i^\beta)| < 2^{-\gamma}$$

whenever $i \in N_\gamma$ and $\alpha, \beta \in \{1, 2, \dots, \gamma\}$. Then define an element $(y_i)_{i\in\mathbb{N}} \in (X_i)_\infty$ such that $y_i = x_i^\gamma$ for $i \in N_\gamma \setminus N_{\gamma+1}$. Now $\bar{x}^\gamma \to \bar{y} := (y_i)_\phi$ as $\gamma \to \infty$. □

If each X_i is a Banach or Hilbert space, then $(X_i)_\phi$ has a canonical Banach or Hilbert space structure, respectively.

The ultralimit $(X)_\phi$ of a constant sequence (X), $X = (X, d, *)$, is also called the *ultracompletion* (or *ultrapower*) of X. It is independent of the choice of the basepoint $*$, and the map that assigns to each $x \in X$ the equivalence class $(x)_\phi$ of the constant sequence (x) is a canonical isometric embedding of X into $(X)_\phi$. Ultrapowers have applications e.g. in Banach space theory, see [Hein], [HeinM].

For a fixed metric space (X, d), a sequence of basepoints $*_i \in X$, and a sequence of scale factors $\lambda_i > 0$ with $\lim_{i\to\infty} \lambda_i = \infty$, the ultralimit $(X_i)_\phi$ of the sequence of rescaled metric spaces $X_i = (X, \frac{1}{\lambda_i} d, *_i)$ is referred to as an *asymptotic cone* of X. This construction has played a significant role in recent proofs of rigidity theorems in the theory of nonpositively curved spaces, see e.g. [KleL].

4.11 Theorem (GH-limit versus ultralimit)
Suppose $(X_i)_{i\in\mathbb{N}}$ is a uniformly precompact and uniformly bounded sequence of metric spaces.

(1) *The ultralimit $(X_i)_\phi$ is isometric to the Gromov-Hausdorff limit of some subsequence (X_{i_k}) of (X_i).*

(2) *If the sequence (X_i) Gromov–Hausdorff converges to some metric space Y, and if each X_i is contained in a pointed metric space $Z_i = (Z_i, *_i)$ with $*_i \in X_i$, then Y isometrically embeds into the ultralimit $(Z_i)_\phi$.*

Proof: (1) We use Theorem 4.3 (Gromov embedding) and assume that each X_i belongs to the set $\mathcal{C}$ of all non-empty compact subsets of some fixed compact metric space Z. Then the map $f\colon (X_i)_\phi \to Z$ that assigns to each class $(x_i)_\phi$ the limit $\lim_\phi x_i$ is a well-defined isometric embedding since

$$d(\lim_\phi x_i, \lim_\phi y_i) = \lim_\phi d(x_i, y_i) = d((x_i)_\phi, (y_i)_\phi).$$

By Theorem 4.4(2) (Blaschke), $(\mathcal{C}, d_H)$ is compact, hence the sequence (X_i) has a unique ϕ-limit $\lim_\phi X_i = Y \in \mathcal{C}$ with respect to d_H. We show that $f((X_i)_\phi)$ coincides with Y; then $(X_i)_\phi$ is isometric to the Gromov-Hausdorff limit of some subsequence of (X_i) as claimed.

Let $y \in Y$. For every i, choose $x_i \in X_i$ such that $d(x_i, y) \le d_H(X_i, Y)$. Since $\lim_\phi X_i = Y$, $\lim_\phi d(x_i, y) \le \lim_\phi d_H(X_i, Y) = 0$ and hence $\lim_\phi x_i = y$. This shows that $Y \subset f((X_i)_\phi)$.

To prove the reverse inclusion, let (x_i) be a sequence with $x_i \in X_i$ for all i. Choose $y_i \in Y$ such that $d(x_i, y_i) \le d_H(X_i, Y)$. Then $d(\lim_\phi x_i, \lim_\phi y_i) = \lim_\phi d(x_i, y_i) \le \lim_\phi d_H(X_i, Y) = 0$, hence $\lim_\phi x_i \in Y$.

(2) Using (1) we see that the completion $\bar{Y}$ of Y is isometric to $(X_i)_\phi$, and $(X_i)_\phi$ canonically embeds into $(Z_i)_\phi$. □

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