PCMI GMT Lecture # 6

Stories and Results

1900 — 1960

Douglas
Besicovitch
L.C. Young
Federer
De Giorgi
Whitney
⋮

1960
① Federer and Fleming "Normal and Integral Currents"
② De Giorgi "Frontiere Orientate di Misura Minima"
③ Reifenberg "solution of the Plateau problem for m dimensional surfaces of varying topological type"

1960 - Present (some holes here !!)

- Federer, Fleming students (68 + 136) Descendants

Allard
(82) Almgren → Taylor, Pitts, Parks ⋮ (18 total)
(13) ← Liu ⋮ (16) ← Hardt (31)
Zeimer

(some numbers are from Mathematics Genealogy project)

- De Giorgi — Ambrosio ... (47) Descendants
Alberti
Braides
Buttazzo
Dal Maso
Lenci

- Leon Simon
- David Preiss
- P Mattila
- P Jones
- Schul
- Lerman
-

⋮

others

GMT : Two main branches

(minimal surfaces)

Calculus of Variations

texts:
- Federers "GMT"
- L. Simon
- F. Morgan
- E. Giusti

Feder, Fleming, De Giorgi

Descendants

(properties of harmonic measures)

Harmonic Analysis

- Mattila } Text
- Guy David
- Semmes
- Jones
- Garnett
- Schul
- Lerman
- Preiss
- Kenig

⋮

names I know ...

GMT as a name comes from Federer 1969

Problems and Results

- Joseph A. Plateau (1801-1883)

  J. Douglas (Fields Medal 1936)

  $f: \mathbb{B}^2 \to \mathbb{R}^3$ $\quad f$ prescribed on $\partial \mathbb{B}^2$

- Federer & Fleming, De Giorgi, Reifenberg

  - existence, some regularity for minimal surfaces.

Examples: (just 4 + a comment... I do not mention lots of deep/important work)

Existence

(1960) Theorem: (compactness & semicontinuity) Suppose $T_1, T_2, \ldots$ are integral currents & $\sup_i (M(T_i) + M(\partial T_i)) < \infty$. Then there is a subsequence converging to an integral current T $T_{i_j} \to T$ (in the integral flat norm). Furthermore Mass (and any elliptic integrand) is lower semicontinuous under this convergence.

$\Rightarrow$ Direct method gives existence

(Fleming) 1962 **Theorem**: A 2-dim area minimizing rectifiable current T in $\mathbb{R}^3$ is a smooth, embedded manifold on the interior.

(Federer) 1970 **Theorem**: An $n-1$ dim area-min rect. current T in $\mathbb{R}^n$ is a smooth embedded manifold on the interior except for a singular set of Hausdorff dimension at most $n-8$.

(Almgren) 1983 **Theorem:** An $n-k$ dimensional area minimizing current in $\mathbb{R}^n$ is a smooth, embedded manifold on the interior except for a singular set of Hausdorff dimension at most $n-k-2$.

This theorem is a result of a paper which for many years circulated as 1600+ mimeographed pages. It is now in book format at about 900 pages.

There are many other results that are worth spending some time with: for example, Jean Taylors results from the mid 1970's definitively explaining the structure of soap bubble clusters. For a nice exposition, see the 1976 article "Geometry of soap films" in Scientific American written by Taylor and Almgren.

Now I move to comments on directions for exploration at the intersection of GMT and data analysis and modeling.

Geometric Analysis and Data

There has been a start: examples include

- extensive TV regularization use and study in image & shape applications
- related Mumford shah work
- Peter Jones Betas; Lerman, Schul, etc.

- Flat norm applications: Clannes, Morgan-Vixie, etc.
- "Varifold" applications: used to allow multiple targets in shapes.
- Non-asymptotic densities: Suato's group, my group. (Manay et al....)
- Fractals: lots of applications.... huge.
- Curvature measures: Adler, Taylor, Worsley, ...
- Diffusion Geometry: Coifman et al.

But this just scratches the surface

Rectifiable sets: exploring, extending the zoo of insights and techniques developed to understand and characterize rectifiable sets

Non-asymptotic objects: densities, measures, tangent cones, normal cones, tangent measures, dimension ...

Currents: normal currents - to represent diffuse, generalized surfaces; stochastic currents - random variables with variables in currents.

Varifolds: very general sets can be represented which nonetheless are analytically tractable.

GMT in metric spaces: This is a relatively new area, developed for its own sake, <u>But</u> it is very relevant for data analyses - data often has a metric but no natural embedding in some $\mathbb{R}^n$.

Driving, Motivating Problems

Probability ⊕ GMT : thinly explored

Example: propagation of uncertainty. computing uncertainty regions, error bars when the state spaces are high dimensional (10^6-10^9, hiring) requires insight & cleverness first - computing power alone will fail.

$\mathbb{R}^3 \oplus \mathbb{R}^\infty$:

Petascale image streams present simultaneous low/high dimensional challenges: images are points in $\mathbb{R}^\infty$, these points represent low dimensional objects, scenes, shapes.

Example: Doing high dimensional metrics using insights into low dimensional structures and processes generating these structures, is a critical component of defensible inference from data.

$N \ll \infty$:

Geometric analytic methods are useful for moving (non-black-box style) from finite data to predictive inferences.

By finite, I mean {finite and less than some known N}, not {finite, but as big as we need}

Example: what can we say about where a boundary is, given 1000 points from inside the boundary?

Bibliography (order of appearance not meaniful!)

(There are holes in this list ...)

GMT and Geometric Analysis

Texts

- Federer 1969: the GMT Bible, not easy to read unnecessarily devoid of illustrations, but very geometric to those so inclined. A must have
- F. Morgan 3rd Ed 2000: Good introduction, maybe the best.
  4th ?
- Leon Simon, 1983: One of my favorites – economical, well written, doesn't try to do everything.
- F. Lin & X. Yang 2003: Good selection of topics, typos can be irritating. more up to date than others
- Krantz & Parks 2008 "Geometric Integration Theory" Nice aspects, some details that are hard to find other places
- H. Whitney 1957, 2005 "Geometric Integration theory" Classic ... much differs than other books above.
- Mattila 1999 "Geometry of sets and measures in Euclidean Spaces: Fractals and Rectifiability" GMT from the harmonic analysis angle ... good book, Measures & rectifiability, like the title says
- Burago & Zalgaller 1988 "Geometric Inequalities" lots of interesting stuff here!

- Morvan 2008 "Generalized Curvatures"
  Interesting content, I haven't read pieces yet, but I do own it.

- Adler & Taylor 2007, "Random Fields and Geometry"
  application of curvature measures to statistics.

- Camillo De Lellis 2008 "Rectifiable sets, Densities and Tangent Measures"
  This is an exposition of Preiss' big paper. Very good, though I have yet to work through it... but it is high on my priority list.

- Giusti 1984 "Minimal Surfaces and Functions of Bounded Variation"
  Excellent exposition of BV functions, sets of finite perimeter and the application to minimal surface problems. This is essentially what De Giorgi did in 1960.

- Evans & Gariepy 1999 "Measure theory and Fine properties of functions"
  Beautiful book: my favorite for the topics it covers, including BV functions & sets of finite perimeter.

- Morel & Solimini 1995 "Variational Methods in Image Segmentation" Nice book, I will work through it when some student wants to: Images & GMT (rectifiable sets, measures) are covered in an intuitively appealing way.

Elias Stein "Singular Integrals and Differentiability properties
1971 of Functions"

Beautiful book .. I plan to teach a seminar out of it soon for my own sake.

Harmonic analysis from a perspective that is attractive to me.

Ambrosio, Fusco, Pallara "Functions of Bounded Variation and
2000 Free Boundary Problems"

BV & SBV functions, and the Mumford Shah functional. Background for Ambrosio's work on existence for Mumford Shah. Fairly popular as intro to BV functions.

There is also books by Giaquinta (2 volume "Cartesian Currents in Calculus of Variations", "Maps into manifolds and currents") as well as Magnani's "Elements of geometric measure theory on sub-riemannian groups".

Papers

- Fred Almgrens Collected Papers
- Federer's papers
- Anything from Camillo De Lellis (Zurich)
- Ambrosio & Kirchheim "Currents in Metric Spaces"
- De Pauw & Hardt "Rectifiable & Flat G chains in metric space"
- cvgmt.sns.it (Pisa website)
- Allard's 1972 paper of Varifolds
- Look at the Bibliography in Frank Morgan's Book and in Mattila's book.