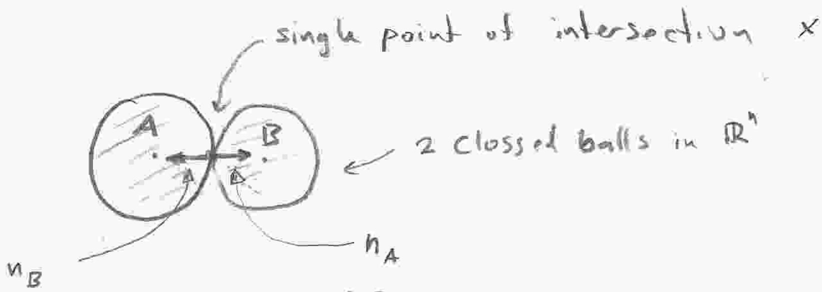
PCMI GMT Lecture #5

4.10

This theorem establishes the properties of $C \subset A \cap B$ in terms of the properties of A & B. A, B and C are assumed closed and C is assumed compact.

(3)

The condition on lines 5 & 6 of the theorem are necessary because of examples like



$\{x\} = C = A \cap B$

$\mathrm{Tan}(A,x) = \{u \mid u \cdot n_A \le 0\}$, $\mathrm{Nor}(A,x) = \{v \mid v = \lambda n_A, \lambda \ge 0\}$

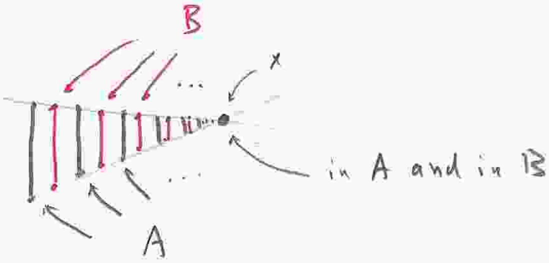
$\mathrm{Tan}(B,x) = \{u \mid u \cdot n_B \le 0\}$, $\mathrm{Nor}(B,x) = \{v \mid v = \lambda n_B, \lambda \ge 0\}$

$\mathrm{Tan}(C,x) = \{0\}$, $\mathrm{Nor}(C,x) = \{v \in \mathbb{R}^n\}$

This violates (3).

In general, it is easy to construct examples $\ni$ A, B & C are closed C is compact and (3) is violated because A & B do not have positive reach.

Example

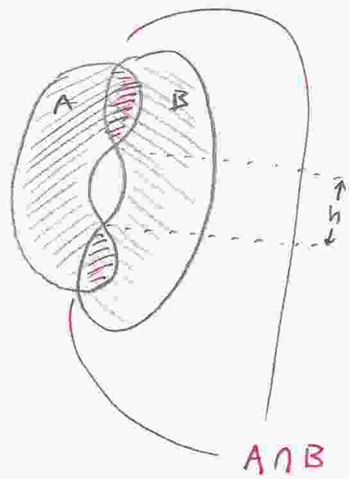


A, B closed, $\{x\} = C = A \cap B$

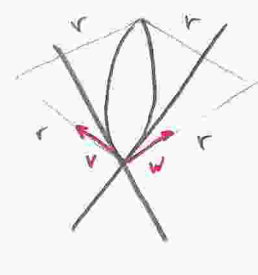
$$\mathrm{Tan}(A,x) = \mathrm{Tan}(B,x) \neq \mathrm{Tan}(C,x)$$

$$\mathrm{Nor}(A,x) = \mathrm{Nor}(B,x) \neq \mathrm{Nor}(C,x)$$

(5) Example shows the necessity of η in this inequality: the basic idea is shown here



$\mathrm{reach}(A \cap B) = \frac{h}{2}$



$$\frac{|v+w|}{|v|+|w|}$$

letting $a = |v|$
$b = |w|$
$c = |v+w|$

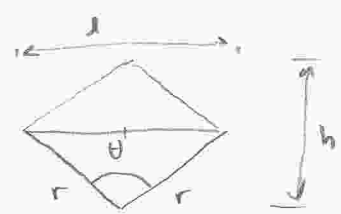
$$\left(\frac{|v+w|}{|v|+|w|}\right)^2 = \frac{a^2+b^2-2ab\cos(\pi-\theta)}{a^2+b^2+2ab}$$

without loss of generality choose $a=1$

$$\frac{1+b^2+2b\cos\theta}{1+b^2+2b}$$

minimizing for fixed θ we get $b=1 \Rightarrow \eta = \sqrt{\frac{1+\cos\theta}{2}}$.

Now



$$\frac{h}{2} = \sqrt{r^2-\left(\frac{l}{2}\right)^2}$$

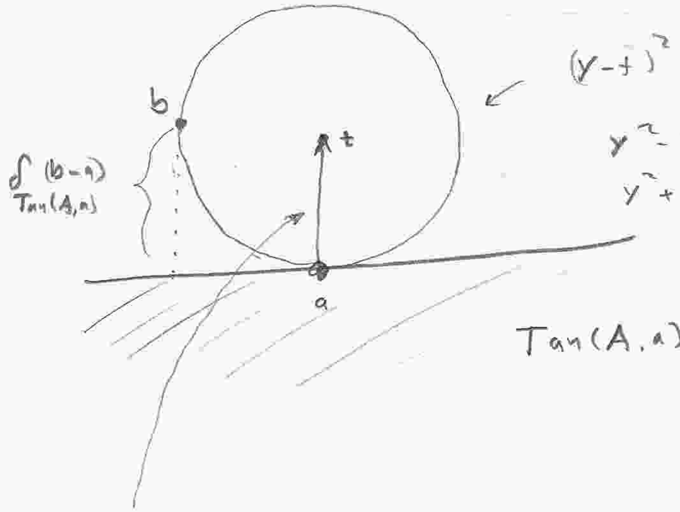
$$l = r\sqrt{2}\sqrt{1-\cos\theta} \quad \text{so} \quad \frac{h}{2} = r\sqrt{\frac{1+\cos\theta}{2}} = r\eta.$$

so we get $\text{reach}(A\cap B) = r\eta$.

<u>Comment</u>: I have not worked through Federers development of this, but I <u>believe</u> that his inequality is <u>not</u> tight, that in fact $\text{reach}(A\cap B) \geq r\eta$ is <u>always</u> <u>true</u>. I know from a conversation with Hal Parks that another piece of "Curvature Measures" states a result which is not tight, not optimal ... this is surprising for Federer.

4.18

$$\left\{ \text{reach}(A) \geq t \right\} \Leftrightarrow \left\{ d_{\text{Tan}(A,a)}(b-a) \leq |b-a|^2/2t \quad \forall\, a,b \in A \right\}$$



$$(y-t)^2 + x^2 = t^2$$

$$y^2 - 2yt + t^2 + x^2 = t^2$$

$$y^2 + x^2 - 2yt = 0 \Rightarrow |b-a|^2 - 2\delta_{Tan(A,a)}(b-a)\,t = 0$$

$$\Downarrow$$

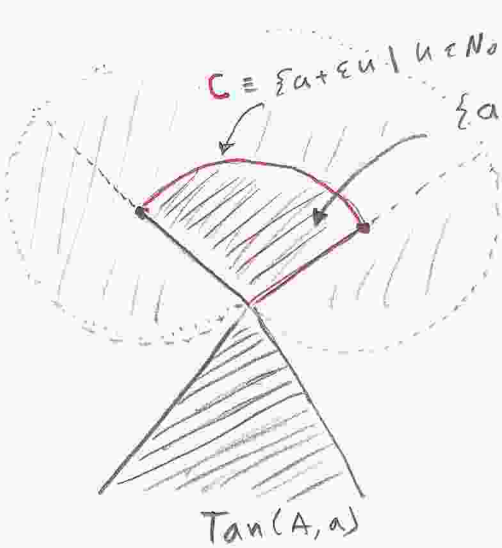
$$\delta_{Tan(A,a)}(b-a) = \frac{|b-a|^2}{2t}$$

$$\Downarrow$$

for b <u>outside</u> interior of circle

$$\delta_{Tan(A,a)}(b-a) \le \frac{|b-a|^2}{2t}$$

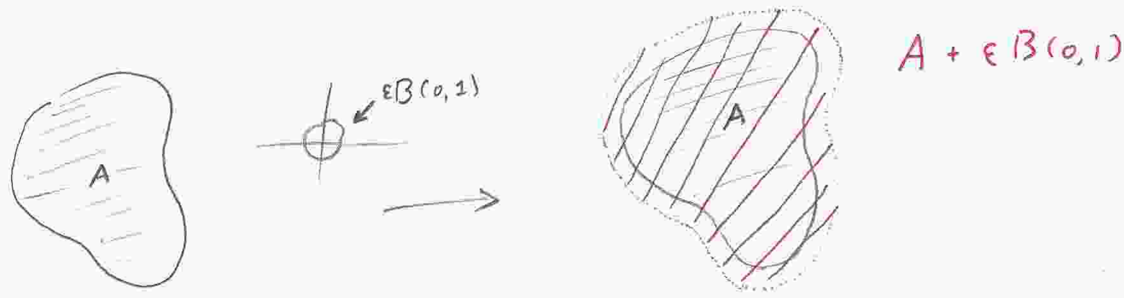
every point $a + \varepsilon v$ for $v \in Nor(A,a)$ al $\varepsilon < t$ has a as it's <u>unique</u> closest point in A. $\Rightarrow$ any point in A must reside <u>on</u> or outside of the circle centered at $a + tu$.



no points $b \in A$ in the interior of the union of the disks of radius t centered on C.

same reasoning works...

Steiner-Minkowski and Curvature Measures



What is $\mathcal{H}^n(A + \varepsilon B)$ in terms of properties of A?

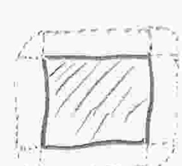
Let's experiment in 2d:

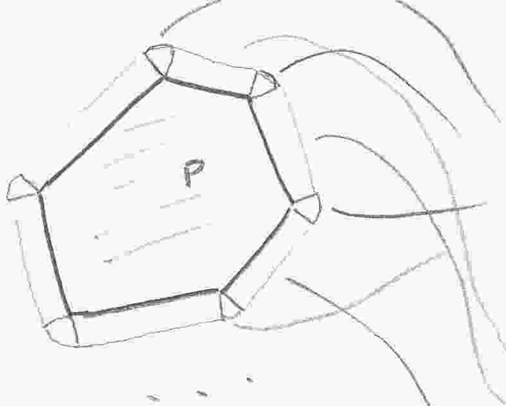
$A = B(0,r)$

$$\longrightarrow \quad \mathcal{H}^2(A + \varepsilon B) = \pi r^2 + \varepsilon 2\pi r + \varepsilon^2 \pi$$

$A =$ solid square

$$\longrightarrow \quad \mathcal{H}^2(A + \varepsilon B) = h^2 + \varepsilon 4h + \varepsilon^2 \pi$$





these areas sum to $\varepsilon^2 \pi$

these areas sum to $\varepsilon \mathcal{H}^1(\partial P)$

so $\mathcal{H}^2(P + \varepsilon B) = \mathcal{H}^2(P) + \varepsilon \mathcal{H}^1(\partial P) + \varepsilon^2 \pi$

In general, as long as $\varepsilon <$ reach (A)

$$\mathcal{H}^2(A + \varepsilon B) = \mathcal{H}^2(A) + \varepsilon \mathcal{H}^1(\partial A) + \varepsilon^2 \pi$$
$$= \mathcal{H}^2(A) + \varepsilon \int_{\partial A} 1 \, d\mathcal{H}^1 + \tfrac{1}{2}\varepsilon^2 \int_{\partial A} \kappa \, d\mathcal{H}^1$$

In the case of polygons we get a generalized κ which is a sequnce of weighted point masses.

Theorem 5.6 (my version)

$$*) \; \mathcal{L}^n(A + \varepsilon B) = \mathcal{L}^n(A) + \varepsilon \int_{\partial A} 1 \, d\mathcal{H}^{n-1} + \frac{\varepsilon^2}{2} \int_{\partial A} \Sigma \kappa_i \, d\mathcal{H}^{n-1} + \dots + \frac{\varepsilon^{m+1}}{m+1} \int_{\partial A} \sum_{\alpha_m} \kappa^{\alpha_m} d\mathcal{H}^{n-1} + \dots + \frac{\varepsilon^n}{n} \int_{\partial A} \kappa_1 \kappa_2 \cdots \kappa_{n-1} d\mathcal{H}^{n-1}$$

$$*) \; \mathcal{L}^n(x \mid \delta_A(x) \le \varepsilon < \text{reach}(A),\ \xi(x) \in K)$$
$$= \mathcal{L}^n(K \cap A) + \varepsilon \int_{K \cap \partial A} \overbrace{1 \, d\mathcal{H}^{n-1}}^{d\Psi_{n-1}} + \frac{\varepsilon^2}{2} \int_{K \cap \partial A} d\Psi_{n-2} + \dots \frac{\varepsilon^n}{n} \int_{K \cap \partial A} d\Psi_0$$

(go to slides of geometric proof for nice sets)

(mention Gauss map if there is time)