PCMI GMT Lecture # 4

We now move to a look at pieces of Federer's 1959 paper "Curvature Measures".

We will look at 3 pieces:

1. coarea formula
2. properties of sets of positive reach
3. Steiner-Minkowski and curvature measures.

Again, the emphasis is intuitive and geometric ... pictures and examples are the emphasis.

Coarea Formula (3.1)

Theorem. If X and Y are separable Riemannian C' manifolds with

$$\dim X = m \geq K = \dim Y$$

and $f : X \to Y$ is Lipschitz, then

$$\int_A Jf(x)\, d\mathcal{H}^m_X = \int_Y \mathcal{H}^{m-K}(A \cap f^{-1}(\{y\}))\, d\mathcal{H}^K_Y$$

whenever A is an $\mathcal{H}^m$ measurable subset of X

$Jf(x) \equiv \sqrt{\det(Df \circ Df^T)}$ where Df is the derivative of f evaluated at x.

and consequently ...

$$\int_X g(x)\, Jf(x)\, d\mathcal{H}^m x = \int_Y \int_{f^{-1}(\{y\})} g(x)\, d\mathcal{H}^{m-k}_x \, d\mathcal{H}^k_y$$

whenever g is an $\mathcal{H}^m$ integrable function on X

Examples:

This is a far reaching generalization of Fubini's theorem:

① choosing $f(x) = x_1$ $(x = (x_1, x_2, \dots, x_n))$ we have, $X = \mathbb{R}^n$, $Y = \mathbb{R}^1$

$$\int g(x)\, d\mathcal{L}^n_x = \int_{\mathbb{R}^1} \int_{\underset{\uparrow\ \mathbb{R}^{n-1}}{x=(y,s)}} g(x)\, d\mathcal{L}^{n-1}_y \, d\mathcal{L}^1_s$$

② Choosing $f(x) = d(x, K)$, any compact $K \ni \mathcal{L}^n(K) = 0$, $x \in \mathbb{R}^n$, $K \subset \mathbb{R}^n$

Then $Jf = 1$ $\mathcal{L}^n$ a.e. in $\mathbb{R}^n$ and we can obtain $\int_{\mathbb{R}^n} g$ by first integrating along the level sets of f and then integrating over the levels.

③ a common $f(x)$ is $d(x, 0)$ giving us spheres

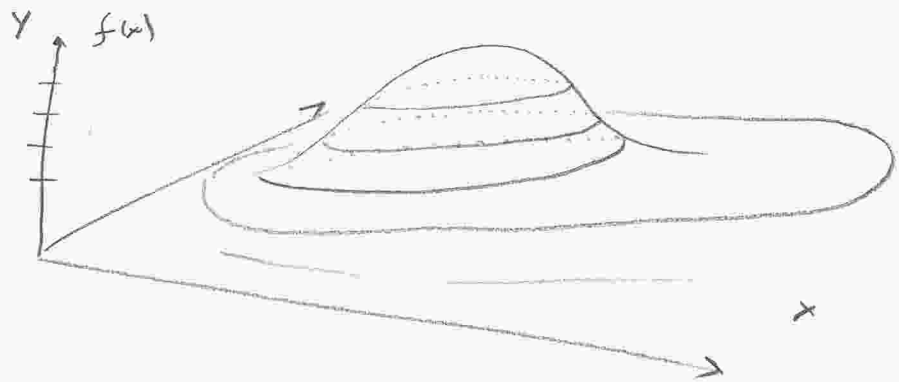
$$\int g(x) = \int_0^\infty \int_{\partial B(0,\rho)} g(x)\, d\mathcal{H}^{n-1}_x \, d\mathcal{L}^1_\rho$$

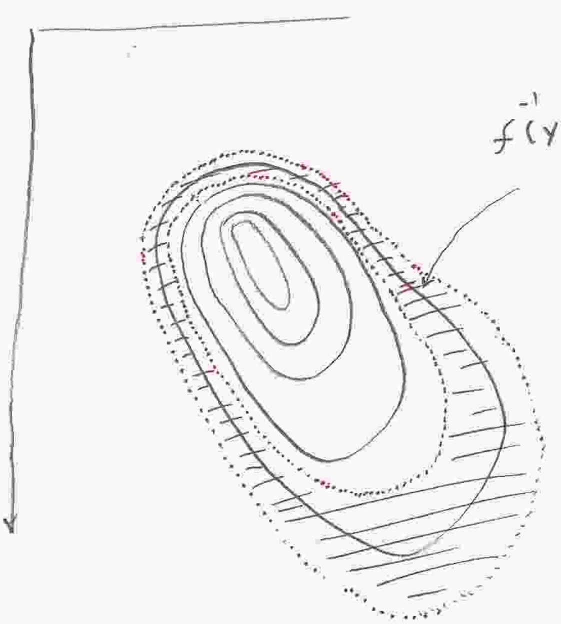
When g is radially symmetric we get

$$\int g \, dx = \int_0^\infty \alpha(n-1)\, \rho^{n-1} g(\rho)\, d\rho$$

(4) repeats from the second lecture: the BV coarea does not follow directly from this theorem. Here f is Lipschitz.

(5) A picture: we look at the case of $X = \mathbb{R}^2$, $Y = \mathbb{R}$ and f is the function





$= f^{-1}((y^*-\varepsilon, y^*+\varepsilon))$

we observe that $\mathcal{H}^1(f^{-1}(\{y^*\}))\cdot 2\varepsilon$

$$\cong \int Jf \, dx$$

the "tube"

... This is the geometric intuition behind the theorem.

(6) notice also that the theorem implies that Flat spots are no problem (no surprise there) and that for almost every y $f^{-1}(\{y\}) \cap B(0,R)$ is a set with finite $\mathcal{H}^{m-k}$ measure.

In fact more is true: the proof of the theorem shows that $f^{-1}\{x\}$ is countably $(\mathcal{H}^{m-k}, m-k)$ rectifiable for $\mathcal{H}^k$ almost all y in Y.

Sets with positive reach (4.8, 4.10, 4.18)

Definitions

- $\delta_A(x) = \text{distance}(x, A) = \inf\{|x-a| : a \in A\}$

- $Unp(A) \equiv x \in E_n$ ($E_n = \mathbb{R}^n$ with Euclidean norm) $\ni$ there exists a unique $a_x \in A$ with $\delta_A(x) = |x-a_x|$.

- $\xi_A : Unp(A) \longrightarrow A$, $\xi_A(x) = a_x$

- $\text{reach}(A, a) \equiv \sup\{r \mid \{x : |x-a| < r\} \subset Unp(A)\}$

- $\text{reach}(A) \equiv \inf\{\text{reach}(A,a) : a \in A\}$

- $\text{Tan}(A, a) \equiv \left\{u \in E_n \;\middle|\; \forall \varepsilon > 0\ \exists b \in A,\ 0 < |b-a| < \varepsilon,\ \left|\frac{b-a}{|b-a|} - \frac{u}{|u|}\right| < \varepsilon\right\} \cup \{0\}$

tangent vectors of A at a. not a measure theoretic concept!

- $\text{Nor}(A, a) \equiv \{v \in E_n \mid v \cdot u \leq 0 \quad \forall u \in \text{Tan}(A,a)\}$

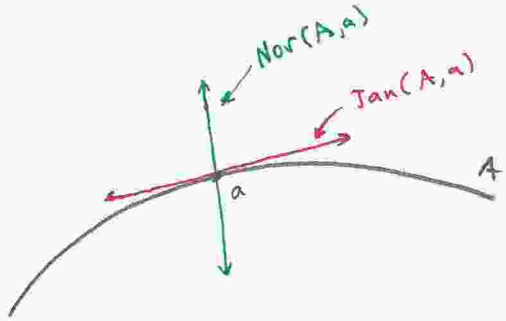
•) $\mathrm{Dual}(S) \equiv \{ v : v \cdot u \le 0 \quad \forall u \in S \}$ and is a closed convex cone.

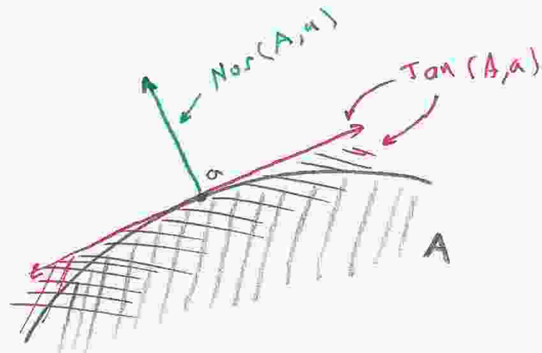
$\mathrm{Dual}(\mathrm{Dual}(S))$ is the smallest closed convex cone containing S.

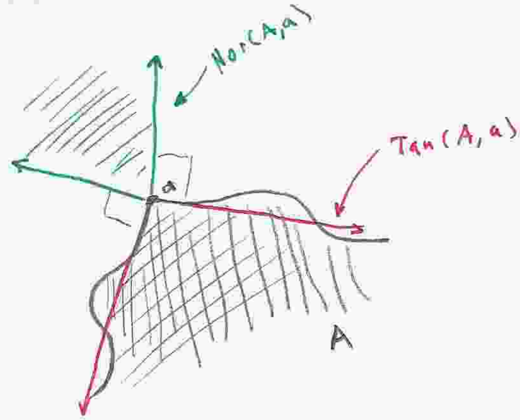
Clearly $\mathrm{Nor}(A,a) = \mathrm{Dual}(\mathrm{Tan}(A,a))$.

$\mathrm{Tan}(A,a)$ is closed and homogeneous but not necessarily convex.

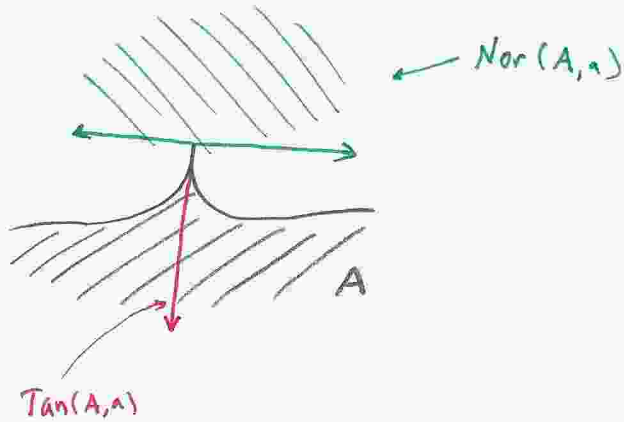
Examples:

① 

② 

③ 

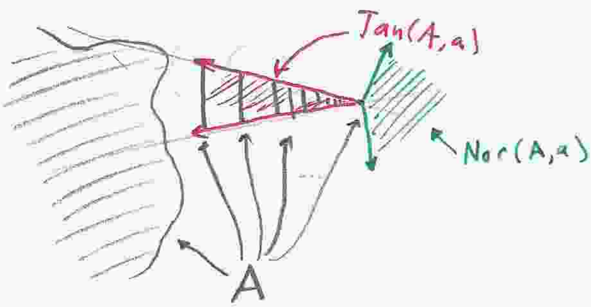
(4)



(5)

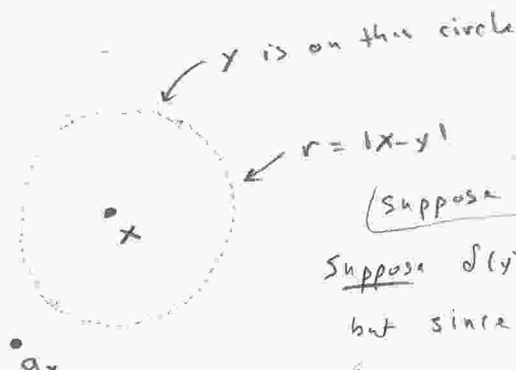
If A is $\{\{x,y\} \ni x \,\&\, y$ are rational$\}$
then $\mathrm{Tan}(A,a) = E_2 \quad \forall\ a \in A$

(6)



Theorem 4.8 (we now look at some pieces)
(project the theorem onto screen)

(1)



First a geometric proof...

$\boxed{\text{Suppose } |x-y| < \delta(y)}$

Suppose $\delta(y) > \delta(x) + |x-y|$
but since y is on the circle
$|y-a_x| \le \delta(x) + |x-y|$

suppose $\delta(y) < \delta(x) - |x-y|$
$\Rightarrow \quad |x-a_y| \le |x-y| + |y-a_y| = |x-y| + \delta(y)$
$< \delta(x) \quad \Rightarrow\Leftarrow$

if $\delta(x) \le |x-y|$

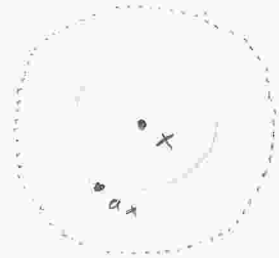
then $|\delta(y)-\delta(x)| > |x-y|$

iff $\delta(y) > |x-y| + \delta(x)$

but on the circle

$|y-a_x| \le |x-y| + \delta(x)$

$\Rightarrow\Leftarrow$.



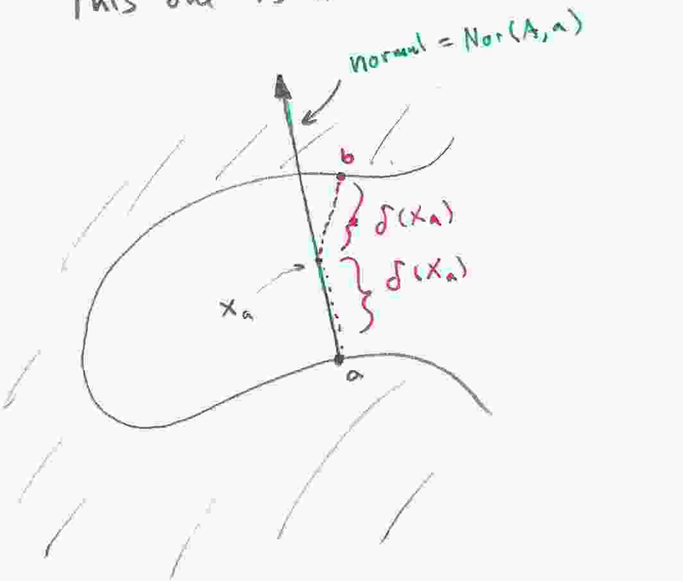
<u>Proof in the paper:</u> choose $a \in A \ni \delta(x) = |x-a|$

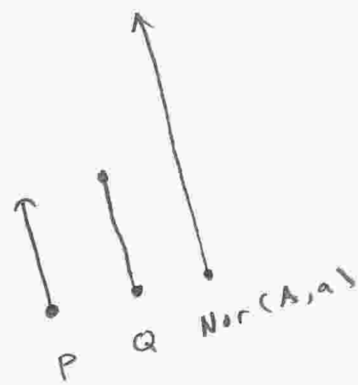
then:

$$\delta(y) - \delta(x) \le |y-a| - |x-a| \le |y-x|$$

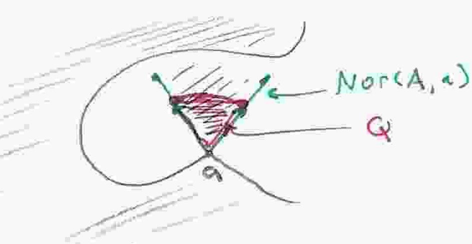
!!

(2) This one is a bit more involved



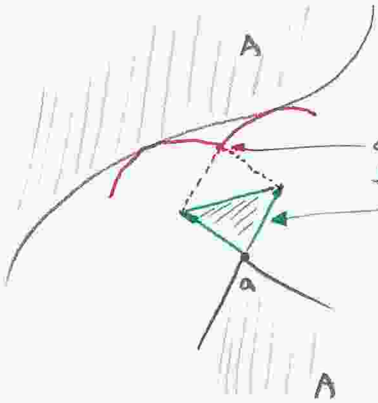


It gets more interests when Nor(A,a) is not one dimensional.



the idea is that is we can go out to $v = \alpha_1 n_1$ on n_1 and $u = \alpha_2 n_2$ on n_2

↓



picture proof in 2D of the convexity of Q.

The actual proof is not hard it starts

$$v \in P \iff |b-v| > |v| \quad \forall \quad b \in A \setminus \{a\}$$

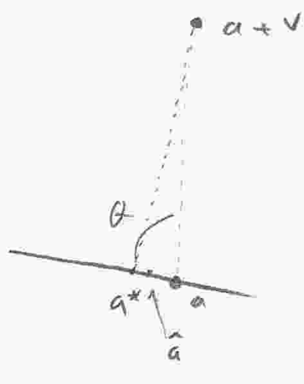
$$v \in Q \iff |b-v| \geq |v| \quad \forall \quad b \in A \setminus \{a\}$$

and then notes that

$$|b-v|^2 - |v|^2 = b \cdot (b-2v)$$

this gives us the convexity of P & Q.

That $P, Q \subset Nor(A,a)$ follows by a simple geometric argument (i.e. choose $v \notin Nor(A,a)$ and assume $\xi(a+v) = a$



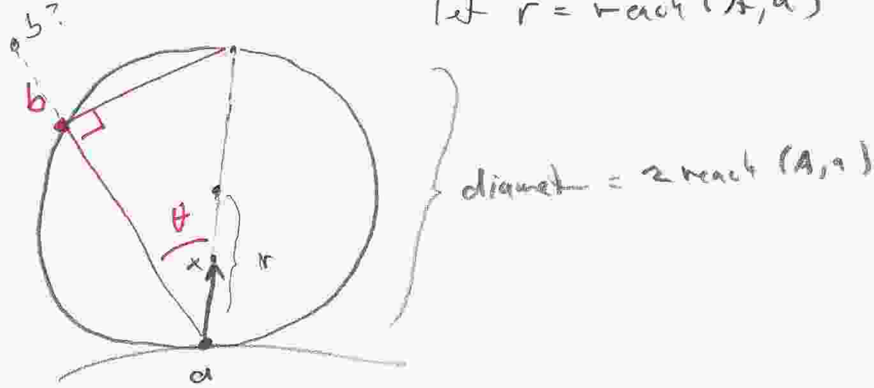
$\theta < \pi/2 \Rightarrow \exists$ a point $\hat{a}$ between a^* & $a \ni$

$$|\hat{a} - (a+v)| < |v| \quad)$$

Skipping ahead to

(7)

let $r = \text{reach}(A, a)$



$$(x-a)\cdot(a-b) \geq -\frac{|a-b|^2\,|x-a|}{2\cdot\text{reach}(A,a)}$$

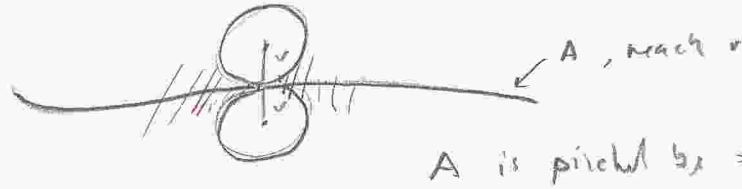
$$\Downarrow$$

$$\underbrace{\frac{x-a}{|x-a|}\cdot\frac{b-a}{|b-a|}}_{\cos\theta} \leq \frac{|b-a|}{2\,\text{reach}(A,a)}$$

$\Rightarrow \quad 2\,\text{reach}(A,a)\cdot\cos\theta \leq |b-a|$

$\Rightarrow$ b is <u>on</u> or <u>outside</u> of circle

<u>Implication</u>



A is pinched by tangent spheres...