PCMI GMT Lecture # 3

continuing with 4.5.9

(15) E is the jump set, so for $W \subset E$

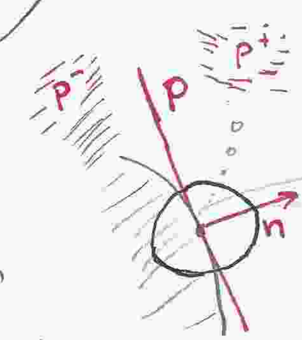
$\|\partial T\|(W)$ = integral of height of jump along W

$$= \int_W (\mu - \lambda)\, d\mathcal{H}^{n-1}$$

(16) & (17) E is an $n-1$ rectifiable current in $\mathbb{R}^n$

in the case that $f = \chi_\Omega$, this means the set $\{x \ni \lambda(x) < \mu(x)\} = \partial^*\Omega$ ← reduced boundary of Ω

= a "nice" set with exterior normals at $\mathcal{H}^{n-1}$ almost every point in E

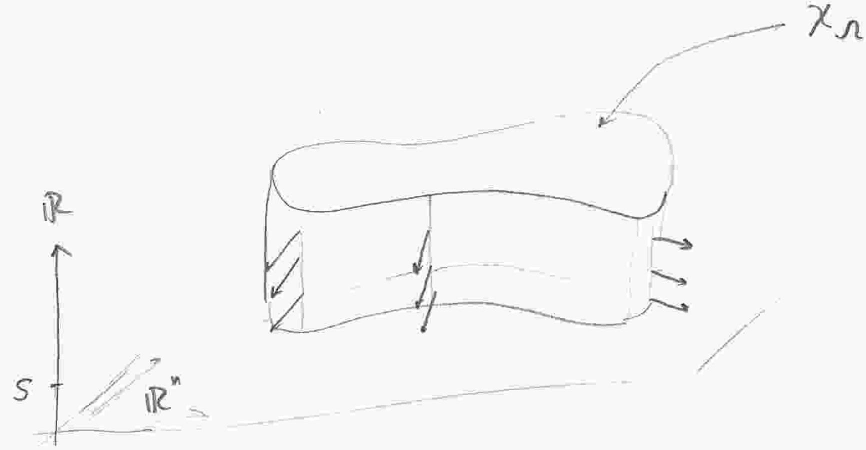


$$\lim_{r \to 0} \frac{\mathcal{L}^n(B_r(x) \cap P^+ \cap A)}{\alpha(n)\, r^n} = 0$$

$$\lim_{r \to 0} \frac{\mathcal{L}^n(B_r(x) \cap P^- \cap A)}{\alpha(n)\, r^n} = 0$$

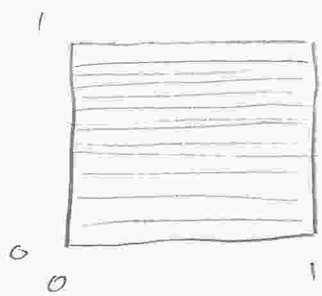
Def: Exterior normal

P the hyperplane with normal n divides Ω locally, measure theoretically into inside & outside



$$n[\underbrace{\overset{\text{underset of } f}{\overset{\downarrow}{G}}, \underbrace{(\vec{b}, s)}_{\text{point in } \mathbb{R}^{n+1}}}] = \underbrace{(\overbrace{n[\{x \mid f(x) \geq s\}, \vec{b}]}^{\text{vector in } \mathbb{R}^n}, 0)}_{\text{vector in } \mathbb{R}^{n+1}}$$

<u>Note:</u> E is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable <u>not</u> $(\mathcal{H}^{n-1}, n-1)$-rectifiable. We can create an example $\ni$ for any compact K $\mathcal{H}^{n-1}(K \cap E) = \infty$.

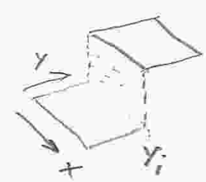


$E = \bigcup \{x, y \mid y \text{ is rational}\}$

To get an f with this jump set, enumerate the rational y, $\{y_1, y_2, y_3, \ldots\}$ and then let f_i be defined by

$$f_i : [0,1]^2 \to [0,1] : (x,y) \to \frac{1}{2^i}\begin{cases} 0 & y < y_i \\ 1 & y_i \leq y \end{cases}$$

f_i



Then $\quad f(x,y) = \sum_i f_i(x,y).$

For this f $\quad \int_{[0,1]^2} |f|\, dx < 1$

and
$$\int_{[0,1]^2} |\nabla f|\, dx = 1 .$$

Remark: we can always use a real multiplicity to turn a countably $(\mathcal{H}^k, k)$-rectifiable set into a real rectifiable current. In fact, Defining $T \equiv \mathbf{E}^n \llcorner f$, for the f just defined, $\partial T = [E]$, $\vec{E} = \vec{e}_1$, $\underset{\substack{\uparrow \\ \text{multiplicity}}}{\rho(x,y)} = \sum \{ \tfrac{1}{2} : \forall i \ni y_i \le y \}$

(19)

The condition

$$\mathcal{L}^n [U(b,\rho) \cap \{x : f(x) > t\}] \le \alpha(n)\, \rho^n/2$$
$$\mathcal{L}^n [U(b,\rho) \cap \{x : f(x) < t\}] \le \alpha(n)\, \rho^n/2$$

is a selection of a <u>median</u> value of f on $\overbrace{U(b,\rho)}^{\text{open ball, radius } \rho \text{, center } b}$.

<u>Notice</u>:

$$\{x : f(x) \ge t\}$$
$$\{x : f(x) \le t\}$$

will <u>not</u> work in the definition.

Example ...

... both expressions yeild sets with measure $> \frac{1}{2}$

of course there is an easy fix

$$\mathcal{L}^n(\ldots \geq t\}] \geq \alpha(n)\, \rho^n/2$$
$$\mathcal{L}^n(\ldots \leq t\}] \geq \alpha(n)\, \rho^n/2 .$$

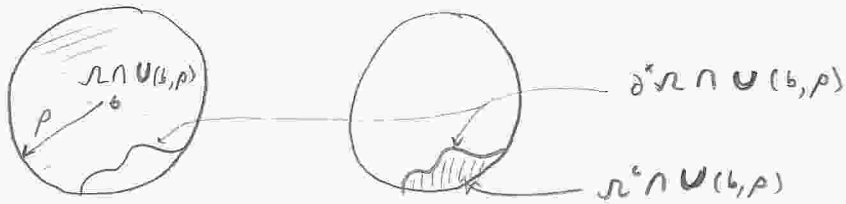
*) $\left(\rho^{-n}\int_{U(b,\rho)} |f-t|^\beta \, d\mathcal{L}^n x\right)^{1/\beta} \leq \sigma\, \rho^{1-n}\, \|\partial T\|(U(b,\rho))$ $\qquad \beta = \frac{n}{n-1}$

is poincaré's inequality for top dimensional normal currents - i.e. Functions of Bounded variation (Everything is local here).

σ is the relative isoperimetric ratio.

**) $\left[\min\left\{\mathcal{L}^n(U(b,\rho)\cap\Omega),\ \mathcal{L}^n(U(b,\rho)\cap\Omega^c\right\}\right]^{1/\beta} \leq \sigma\ \mathcal{H}^{n-1}(\partial^*\Omega \cap U(b,\rho))$

relative isoperimetric constant.



without the "min" we cannot get a relative isop. ratio.

*) has **) as a special case: let f be the characteristic function of Ω, χ_Ω.

Notice that t then automatically makes the left hand side of *) into

$$\left(\rho^{-n}\min\left\{\mathcal{L}^n(U(b,\rho)\cap\Omega),\ \mathcal{L}^n(U(b,\rho)\cap\Omega^c\right)^{1/\beta}$$

and, after factoring out the ρ^{1-n} from both sides we get **.

From (15) and the fact that $\|\partial T\|$ is locally finite we have that $\|\partial T\|$ is Radon. From this we know that

$$\lim_{r \to 0} \frac{\|\partial T\|(B(b,\rho))}{\alpha(n-1)\rho^{n-1}} = 0$$

For $\mathcal{H}^{n-1}$ a.e $b \in \mathbb{R}^n \setminus E$. For such b then,

we have $\Theta^{*n-1}(\|\partial T\|, b) = \Theta_*^{n-1}(\|\partial T\|, b) = \Theta^{n-1}(\|\partial T\|, b) = 0$

and

$$\lim_{\rho \to 0} \int_{U(b,\rho)} |f(x) - t|^{\beta} \, d\mathcal{L}^n x = 0$$

But t depends both on b and on ρ.

(20) and (21) are two inequalities in this direction

in 20 $\lambda(b) = \mu(b) = F(b)$ and F is approx. cont. there. <u>But</u> we are not gauranteed a 0 on the R.H.S. This seems a bit stange at first but ... example

1. iF f is essentially bounded then

$$\rho^{-n} \int_{U(b,\rho)} |f(x) - F(b)|^{\beta} \, d\mathcal{L}^n x \Rightarrow 0$$

as $\rho \to 0$ (By the definition of λ, μ and F)

2. Note that if $b = 0$, $n = 2$ and f is radially symmetric with

$$f(r) = \begin{cases} \frac{1}{\sqrt{r}} & 0 \le r \le L \\ 0 & r > L \end{cases}$$

$\int |f| < \infty$ and $\int |\nabla f| < \infty$

but $F(0) = \infty \Rightarrow$ LHS of the inequality in (20) is ∞. computes:

$$\int_0^\rho \int_0^{2\pi} \frac{1/2}{r^{3/2}} \, d\theta \, dr = \pi \int_0^\rho r^{-1/2} \, dr = 2\pi r^{1/2} \Big|_0^\rho$$

$$= 2\pi \rho^{1/2}$$

$$\Rightarrow \quad \text{RHS} = \sigma \alpha(1) \limsup_{\rho \to 0} \frac{2\pi\rho^{1/2}}{\alpha(1)\rho^1} = \sigma \cdot \infty$$

Putting everything together

1. $\lambda = \mu = F$ on $\mathbb{R}^n \setminus E$
2. $\theta^{n-1}(\|\partial T\|, b) = 0$ $\mathcal{H}^{n-1}$ a.e. in $\mathbb{R}^n \setminus E$
3. (20) (notice that (20) was an everywhere not an a.e. statement)

$\Downarrow$

(21)

(22) measure theoretic normal again.

(29) and (30) when comparing these similar sets of conditions, 29-III and 30-V is one place to see what is different between (29) & (30).

(29) characterizes the jump set part of $\|\partial T\|$ and (30) characterized the cantor like part of $\|\partial T\|$. These two pieces contain all of $\|\partial T\|$ that is singular w.r.t $\mathcal{L}^n$.