PCMI GMT Lecture # 2

Durring this talk, I will have the projector on, showing the various pieces of 4.5.9 on the screen as I decode them for the audience

Federer 4.5.9: properties of n-dimensional normal currents in $\mathbb{R}^n$.

① Explain T, λ, μ, F, G, S, C with pictures

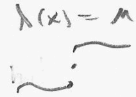
- $T = E^n \llcorner f \in N_n^{loc}(\mathbb{R}^n)$

  $\Rightarrow \int_U |f| < \infty \quad \int_U |\nabla f| < \infty$ for bounded $U \subset \mathbb{R}^n$

- $\lambda(x) \Rightarrow \lambda(x) = \sup\{ l \mid \lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap \{x \mid f(x) < l\})}{\alpha(n) r^n} = 0 \}$

- $\mu(x) \Rightarrow \mu(x) = \inf\{ u \mid \lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap \{x \mid f(x) > u\})}{\alpha(n) r^n} = 0 \}$

- $F(x) = \begin{cases} \lambda(x) = \mu(x) & \text{if } x \text{ is a point of approximate continuity} \\ & \text{half way up or down the jump at any point of the jump set} \end{cases}$

- $G \Rightarrow$ Closure of the set of points below the graph of f in $\mathbb{R}^{n+1}$

- $S \Rightarrow$ The integral current with density 1 supported by the reduced boundary of G.

- $C \Rightarrow$ the reduced boundary of G

- $E \Rightarrow$ the jump set of f ... all the points that are measure theoretically discontinuous.

② parts (1).

f

locally Lipschitz $\Rightarrow$ continuous so $\lambda(x) = \mu(x) = F(x)$

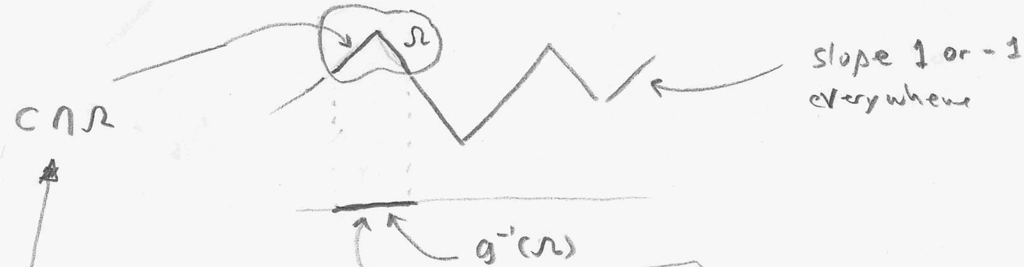
$\{\text{graph of } g\} = C$, $S = [C]$ is the push forward of E^n the current $\mathcal{L}^n \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n$, the current correspond to an oriented $\mathbb{R}^n$.

$\|S\|$ (sometimes called the total variation measure) is the push forward of the measure

$$\mathcal{L}^n \llcorner (1+|Df|^2)^{1/2}$$

picture

For $\Omega \subset \mathbb{R}^{n+1}$, $\|S\|(\Omega) = \int_{g^{-1}(\Omega)} (1+|Df|^2)^{1/2} \, d\mathcal{L}^n$



integrate $(1+|Df|^2)^{1/2} = \sqrt{2}$ over

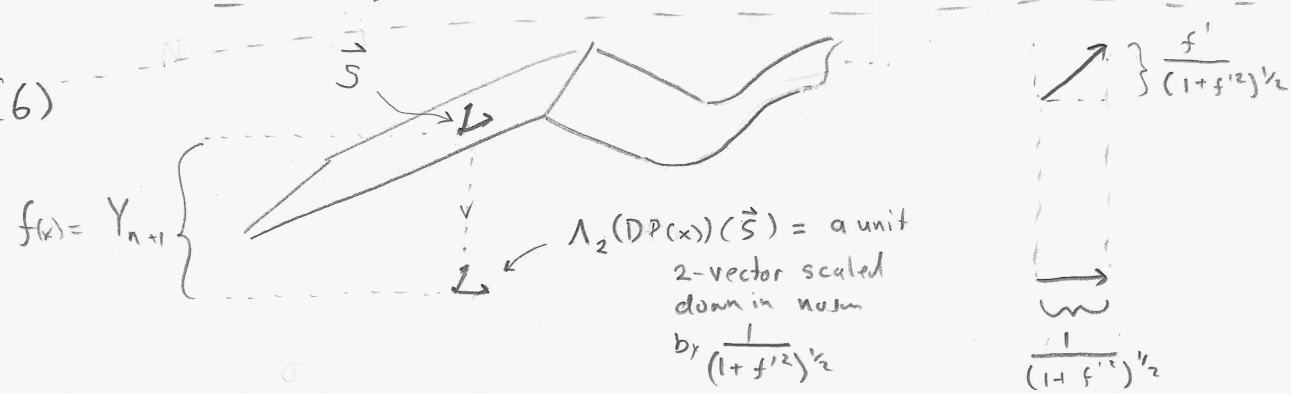
to get length of this

(4) S is an n-dimensional locally rectifiable current. Notice $\partial S = 0$ since S is itself a boundary.. (This is significant... we have dropped the Lip assumption.)

(5) $\|S\|$ is simply $\mathcal{H}^n \llcorner C$, n-dimensional Hausdorff measure restricted to C.

Def: $\mu \llcorner \Sigma(\Omega) \equiv \mu(\Sigma \cap \Omega)$

(6)



Example

Suppose $\phi = h(x)\, dx_1 \wedge dx_2$ $\quad$ $h(x)$ smooth, compact support

then $$\int \langle \vec{T}, \phi \rangle \, d\|T\| = \int_{\mathbb{R}^2} f h \, dx$$
$$= \int \gamma_3 \langle \vec{S}, P^\# \phi \rangle \, d\|S\|$$
$$= \int_S \frac{h(P(y))}{\sqrt{1 + f'(P(y))^2}} \, d\mathcal{H}^n y$$

$$E^n(\phi) = E^n \llcorner 1 = S(\gamma_{n+1} \wedge P^\# \phi)$$ and $\gamma_{n+1} = 1$ on all of S
$$= S(P^\# \phi)$$

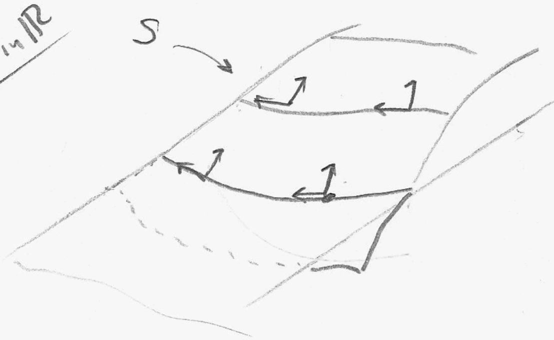
(10) (first part)

$$\|\partial T\| = P_\# \|S \llcorner D\gamma_{n+1}\|$$

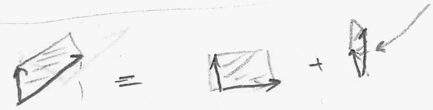
That is,

$$\|\partial T\|(\Omega) = \|S \llcorner D\gamma_{n+1}\|(P^{-1}(\Omega))$$

Example in $\mathbb{R}^2$



in $\mathbb{R}^2$



$\sup_v \langle \vec{S}, D\gamma_3 \wedge Dv \rangle$ is assumed when v is tangent to level set of f.

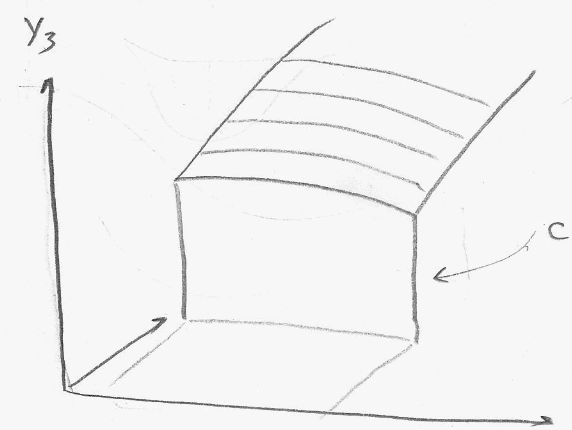
For this v, $\langle \vec{S}, D\gamma_3 \wedge Dv \rangle$
$$= \frac{|\nabla f|}{\sqrt{1 + |\nabla f|^2}}$$
if ∇f is finite and
$= 1$ if we are at a jump set point.

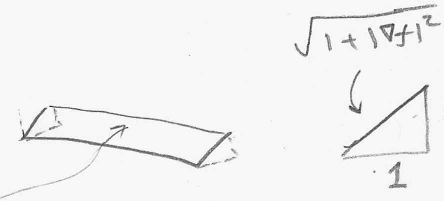
This implies that

$$\|\partial T\|(\Omega) = \int_{p^{-1}(\Omega)\cap C} \frac{|\nabla f|}{\sqrt{1+|\nabla f|^2}}\, d\mathcal{H}^n = \int_\Omega |\nabla f|\, d\mathcal{L}^n$$

with the convention that at jumps the integrand is 1.

easily turns into the BV version of the coarea formula.





integrate $\frac{|\nabla f|}{\sqrt{1+|\nabla f|^2}}$ over this strip gives us the area of the vertical strip.

= length of level set times dy_3

$\Rightarrow$ In the vertical case, the same is obviously true.

Suggests $\Rightarrow$ $\|\partial T\|(\Omega) = \int_\Omega |\nabla f|\, d\mathcal{L}^n = \int_\Omega \mathcal{H}^{n-1}(\Omega_s)\, ds$

(But this is not exactly correct)

where $\Omega_y \equiv \partial\{x \mid f(x) \geq s\} \cap \Omega$

Being a bit more careful, we see that

$$\|\partial T\|(\Omega) = \int_\Omega \mathcal{H}^{n-1}(\Omega_s)\, ds$$

where $\Omega_s = \{x \mid \lambda(x) \le s \le \mu(x)\} \cap \Omega$

But this is just what the next part we consider says.

(14)

$$\|\partial T\|(k) = \iint_{\{x : \lambda(x) \le s \le \mu(x)\}} k \, d\mathcal{H}^{n-1} d\mathcal{L}^1 s$$

where k is some $\bar{\mathbb{R}}$ valued $\|\partial T\|$ integrable function.

Specializing this to $k = \chi_E$,

$$\|\partial T\|(\Omega) = \iint_{\Omega_s} d\mathcal{H}^{n-1} d\mathcal{L}^1 s$$

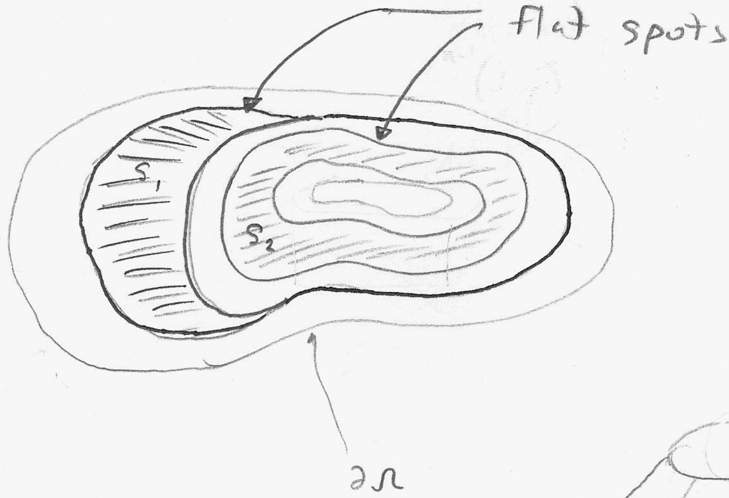
where $\Omega_s = \{x \mid \lambda(x) \le s \le \mu(x)\} \cap \Omega$

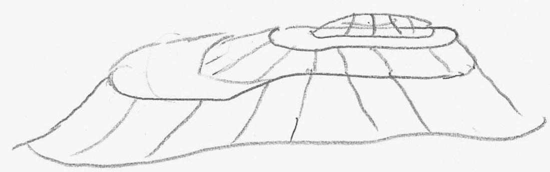
Comparing this with

$$= \mathcal{H}^{n-1}(\partial^* \{x \in \Omega \mid f(x) > s\})$$

$$\|\partial T\|(\Omega) = \int_\Omega |\nabla f| \, dx = \int_{-\infty}^{\infty} \mathrm{Per}(\{x \in \Omega \mid f(x) > s\}) \, ds$$

We see a difference: ... The sets we are measuring with $\mathcal{H}^{n-1}$ at each level differ by flat spots.





notice that for the s_1 level, $\Omega_{s_1} = \{x \mid \lambda(x) \le s_1 \le \mu(x)\} \cap \Omega$ includes the whole flat region whereas $\partial^*\{x \in \Omega \mid f(x) > s_1\}$ does not

$$\Omega_{s_1} = \quad + \quad +$$

$$\partial^*\{x \in \Omega \mid f(x) > s_1\} = \quad +$$

Differences Between $\{\Omega_{s_1}, \partial^*\{x \in \Omega \mid f(x) > s_1\}$

1. notice that the flat spots in $\Omega_{s_1} \to \mathcal{H}^1(\Omega_{s_1}) = \infty$

2. $\mu(x) < s \Rightarrow x$ is in the measure theoretic exterior of $\{x \mid f(x) > s\}$.
$s < \lambda(x) \Rightarrow x$ is in the measure theoretic interior of $\{x \mid f(x) > s\}$.

$$\Rightarrow \partial_*\{x \mid f(x) > s\} \subset \{x \mid \lambda(x) \le s \le \mu(x)\}$$

3. one corollary of (14) is that

$$0 = \int \mathcal{H}^{n-1}(\{x \mid \lambda(x) \le s \le \mu(x)\} \cap \Omega \setminus \partial_*\{x \in \Omega \mid f(x) > s\})\, ds$$