PCMI GMT Lecture #1

In the following 6 lectures, I will give a taste of geometric measure theory, as opposed to an introduction to the subject.

I have chosen to do this through a selective and highly intuitive tour of 2 pieces of the now classical works of late Herbert Federer.

After this first lecture devoted to fundamental definitons and concepts, I will give two lectures explaining pieces of the famous theorem 4.5.9 in Federer's text "Geometric Measure Theory", followed by two more lectures devoted to Federer's 1959 paper "Curvature Measures". The sixth and final lecture will give a bit of history, tell some anecdotes, and touch on the nature of regularity results, lightly.

Currents

The central object of study for the first 3 lectures is the current. To define this we must introduce k-forms ad k-vectors.

simple k-vector

$\boxed{\text{k-vector}}$ $e_1 \wedge e_2 \wedge \dots \wedge e_k$ alternating

intuitively: think of the k-plane spanned by $\{e_1, e_2, \dots, e_k\}$.

- Examples
- properties

- $a_1 e_1 \wedge e_2 + a_2 e_1 \wedge e_3 + a_3 e_2 \wedge e_3$ general 2 vector in $\mathbb{R}^3$
- $e_1 \wedge e_2 = -e_2 \wedge e_1 \Rightarrow e_i \wedge e_i = 0$
- linear in each component

Basis: k-vectors in $\mathbb{R}^n$ have $\binom{n}{k}$ basis vectors

K-Forms

Dual to k-vectors

$$dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$$

$$\langle e_1 \wedge e_2 , dx_1 \wedge dx_2 \rangle = 1$$
$$\langle e_1 \wedge e_3 , dx_1 \wedge dx_2 \rangle = 0$$
$$\langle e_2 \wedge e_3 , dx_1 \wedge dx_2 \rangle = 0$$
$$\vdots$$

Integration

given a set M, a measure μ on M, and a k-vectorfield ξ on M, we define a linear operator on k-forms ϕ by

$$\int_M \langle \xi , \phi \rangle \, d\mu$$

<u>Definition:</u> $\mathcal{D}_K$ = space of continuous linear functionals on the space of smooth, compactly supported k-forms. (we denote this space of smooth, compactly supported k forms by $\mathcal{D}^k$)

Hausdorff Measure

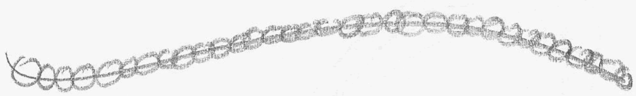
$$\mathcal{H}^k(E) \equiv \lim_{r \to 0} \left(\inf_{\mathcal{F}} \sum_{F_i \in \mathcal{F}} \alpha(k) \left(\frac{\mathrm{diam}(F_i)}{2} \right)^k \right)$$

where $\mathcal{F}$ satisfies $\bigcup_{F \in \mathcal{F}} F \supset E$, $\mathrm{diam}(F) \leq r$

Facts: • in $\mathbb{R}^n$ $\quad \mathcal{H}^n = \mathcal{L}^n$

• $\mathcal{H}^k$ is Borel regular

• if $\mathcal{H}^k(E) < \infty$ $\quad \mathcal{H}^k \llcorner E$ is Radon



Rectifiable Sets

E is $(\mathcal{H}^k, k)$-rectifiable if $E \subset \bigcup_i f_i(E_i)$ where $f_i : \mathbb{R}^k \to \mathbb{R}^n$ are Lipschitz, $E_i \subset \mathbb{R}^k$, and $\mathcal{H}^k(E) < \infty$.

Fact: E $(\mathcal{H}^k, k)$-rectifiable $\Rightarrow$ $E \subset f_i(E_i) \cup N$

f_i C^1, $E_i \subset \mathbb{R}^k$ and $\mathcal{H}^k(N) = 0$

Fact: E $(\mathcal{H}^k, k)$-rectifiable $\Rightarrow$ $\mathcal{H}^k$ a.e. E has approximate tangent planes

$\exists$ k-plane $P \ni \lim_{\lambda \downarrow 0} \int_{\eta_{x,\lambda}(E)} f(y)\, d\mathcal{H}^k = \int_P f(y)\, d\mathcal{H}^k$

f test function

Rectifiable Currents

Let E be $(\mathcal{H}^k, k)$-rectifiable. Define $\vec{E}$ to be a k-vector tangent to E $\mathcal{H}^k$ a.e. (Notice we have to choose 'orientations')

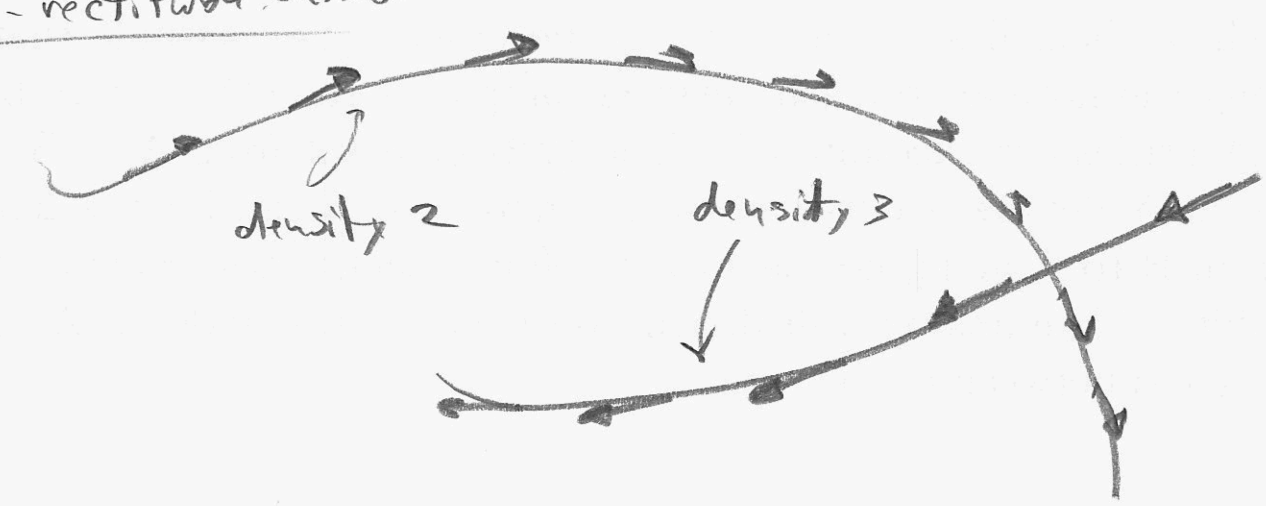
$$\text{Define} \quad [E](\phi) = \int_E \langle \vec{E}, \phi \rangle \, d\mathcal{H}^k$$

more generally

$$[E](\phi) = \int_E \langle \vec{E}, \phi \rangle \, \rho(x) \, d\mathcal{H}^k$$

where $\rho(x)$ is an integer valued function on E.

a 1-rectifiable current



Boundary

k-current

$$\partial[E](\phi) \equiv [E](d\phi)$$

k-1-form

(Example $\phi = f_1 \, dx_1 \wedge dx_2 + f_2 \, dx_3 \wedge dx_1 + f_3 \, dx_2 \wedge dx_3$)

$d\phi = (f_{1,x_1} + f_{2,x_2} + f_{3,x_3}) \, dx_1 \wedge dx_2 \wedge dx_3 = \operatorname{div} f \, dV$)

Mass

Skipping details here

$$M(E) \equiv \sup\{E(\phi) \mid \|\phi\| \le 1\}$$
$$= \mathcal{H}^k(\text{support } E) \text{ when } E \text{ rectifiable.}$$

Integral Currents

T and ∂T rectifiable

Normal currents

$$M(T) < \infty \text{ and } M(\partial T) < \infty$$

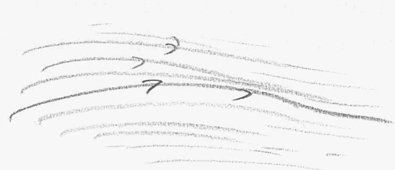
$\Rightarrow$ T & ∂T representable by integration

$$T(\phi) = \int \langle \vec{T}, \phi \rangle \, d\|T\|_x$$
$$\partial T(\phi) = \int \langle \vec{\partial T}, \phi \rangle \, d\|\partial T\|_x$$

<u>Note:</u> ① the support of Normal k-currents <u>need not</u> be k dimensional.

②

1-rectifiable T



smoothed T is normal but <u>not</u> rectifiable

smoothing = convolution with a translation operator.

Push Forwards and Pullbacks

push ξ

ξ = a simple k-vector $v_1 \wedge v_2 \wedge \cdots \wedge v_k$

$f : \mathbb{R}^k \longrightarrow \mathbb{R}^n$, f smooth

Define push forward of ξ in $\Lambda_k \mathbb{R}^n$ at x to be

$$[\Lambda_m(Df(x))](\xi) = (Df(x)(v_1) \wedge Df(x)(v_2) \wedge \cdots \wedge Df(x)(v_k))$$

This map extends linearly to all m-vectors

pull ϕ

for any k-form in $\mathbb{R}^n$, $f^{\#}\phi$ (pullback) in $\mathbb{R}^k$ is given by

$$\langle \xi, f^{\#}\phi(x) \rangle = \langle [\Lambda_m(Df(x))](\xi), \phi(f(x)) \rangle$$

Push forward of a Current $f_{\#}T$

$$(f_{\#}T)(\phi) \equiv T(f^{\#}\phi)$$

Remark: In the case of rectifiable currents with density 1, we simply get that $f_{\#}[E] = [fE]$ ... that is (modulo keeping track of orientations) all we need to do is map the support of the current forward.

References

H. Federer "Geometric Measure theory" 1969

L. Simon "Lectures on Geometric Measure Theory" 1983

F. Morgan "Geometric Measure Theory: A Beginners Guide" 3rd Ed 2000

Comment: Since oriented k-manifolds can be thought of as linear operators on test k-forms, currents are, in a very precise way, simply generalized oriented manifolds.