for $\phi \in \mathscr{D}^n(\mathbf{R}^n)$, and

$$(\partial T)\,\psi = \lim_{j\to\infty} (\partial T_j)\,\psi = [(\partial S) \llcorner Y_{n+1} - S \llcorner D Y_{n+1}]\, p^{\#}\psi$$

for $\psi \in \mathscr{D}^{n-1}(\mathbf{R}^n)$; clearly T is a locally normal current. We also see from 4.3.2(1) that, for $\phi \in \mathscr{D}^n(\mathbf{R}^n)$,

$$T(\phi) = [S \llcorner p^{\#}\phi](Y_{n+1}) = \int \langle e_1, \ldots, e_n, \phi(x)\rangle \cdot \langle S, p, x\rangle (Y_{n+1})\, d\mathscr{L}^n x.$$

4.5.9. Theorem. *If f is a real valued $\mathscr{L}^n$ measurable function such that*

$$T = \mathbf{E}^n \llcorner f \in \mathbf{N}_n^{\mathrm{loc}}(\mathbf{R}^n)$$

and if

$$\lambda(x) = (\mathscr{L}^n)\ \mathrm{ap}\liminf_{z\to x} f(z) \in \overline{R} \ \ \textit{for}\ \ x \in \mathbf{R}^n,$$

$$\mu(x) = (\mathscr{L}^n)\ \mathrm{ap}\limsup_{z\to x} f(z) \in \overline{R} \ \ \textit{for}\ \ x \in \mathbf{R}^n,$$

$$F(x) = [\lambda(x) + \mu(x)]/2 \ \ \textit{for}\ \ x \in \mathbf{R}^n,$$

$$G = \mathbf{R}^{n+1} \cap \{y : \mu(y_1, \ldots, y_n) \geq y_{n+1}\}, \qquad S = (-1)^n\, \partial(\mathbf{E}^{n+1} \llcorner G),$$

$$C = \mathbf{R}^{n+1} \cap \{y : \lambda(y_1, \ldots, y_n) \leq y_{n+1} \leq \mu(y_1, \ldots, y_n)\},$$

$$E = \mathbf{R}^n \cap \{x : \lambda(x) < \mu(x)\},$$

then $\lambda, \mu, F, G, S, C, E$ are uniquely determined by T (because f is $\mathscr{L}^n$ almost determined by T) and the following thirty-one statements hold:

(1) *If $f : \mathbf{R}^n \to \mathbf{R}$ is locally Lipschitzian and*

$$g : \mathbf{R}^n \to \mathbf{R}^{n+1}, \qquad g(x) = (x_1, \ldots, x_n, f(x)) \ \ \textit{for}\ \ x \in \mathbf{R}^n,$$

then g is locally Lipschitzian and

$$S = g_{\#}\mathbf{E}^n, \qquad \|S\| = g_{\#}[\mathscr{L}^n \llcorner (1 + |Df|^2)^{\frac{1}{2}}].$$

(2) *λ, μ, F are Borel functions.*

(3) *For $\mathscr{H}^{n-1}$ almost all x in $\mathbf{R}^n$, $-\infty < \lambda(x) \leq \mu(x) < \infty$.*

(4) *$S \in \mathscr{R}_n^{\mathrm{loc}}(\mathbf{R}^{n+1})$.*

(5) *$\|S\| = \mathscr{H}^n \llcorner C$.*

(6) *For $\phi \in \mathscr{D}^n(\mathbf{R}^n)$, $T(\phi) = S(Y_{n+1} \wedge p^{\#}\phi)$ and $\mathbf{E}^n(\phi) = S(p^{\#}\phi)$.*

(7) *For $\psi \in \mathscr{D}^{n-1}(\mathbf{R}^n)$, $(\partial T)\,\psi = -S(DY_{n+1} \wedge p^{\#}\psi)$.*

(8) $\|T\| = p_{\#}\|S \llcorner DY_1 \wedge \cdots \wedge DY_n \wedge Y_{n+1}\|$.

(9) $\mathscr{L}^n = p_{\#}\|S \llcorner DY_1 \wedge \cdots \wedge DY_n\|$.

(10) *$\|\partial T\| = p_{\#}\|S \llcorner DY_{n+1}\|$ and, for $i \in \{1, \ldots, n\}$,*

$$\|(\partial T) \llcorner DX_1 \wedge \cdots \wedge DX_{i-1} \wedge DX_{i+1} \wedge \cdots \wedge DX_n\|$$
$$= p_{\#}\|S \llcorner DY_1 \wedge \cdots \wedge DY_{i-1} \wedge DY_{i+1} \wedge \cdots \wedge DY_{n+1}\|.$$

(11) $\mathscr{L}^n + \|\partial T\| \geq p_{\#}\|S\|$.

(12) *For $\mathscr{L}^1$ almost all real numbers s,*

$$p_\# \langle S, Y_{n+1}, s\rangle = -\partial[\mathbf{E}^n \llcorner \{x: f(x)\geq s\}] \in \mathscr{R}^{\mathrm{loc}}_{n-1}(\mathbf{R}^n).$$

(13) $\partial T = \int \partial[\mathbf{E}^n \llcorner \{x: f(x)\geq s\}]\, d\mathscr{L}^1 s$ *and*

$$\|\partial T\| = \int \|\partial[\mathbf{E}^n \llcorner \{x: f(x)\geq s\}]\|\, d\mathscr{L}^1 s.$$

(14) *For every $\|\partial T\|$ integrable $\overline{R}$ valued function k,*

$$\|\partial T\|(k) = \iint_{\{x:\, \lambda(x)\leq s\leq \mu(x)\}} k\, d\mathscr{H}^{n-1}\, d\mathscr{L}^1 s.$$

(15) *For every Borel subset W of E,*

$$\|\partial T\|(W) = \mathscr{H}^n[C\cap p^{-1}(W)] = \int_W (\mu-\lambda)\, d\mathscr{H}^{n-1}.$$

(16) *E is countably $(\mathscr{H}^{n-1}, n-1)$ rectifiable.*

(17) *For $\mathscr{H}^{n-1}$ almost all b in E there exists $u\in\mathbf{S}^{n-1}$ such that*

$$\mathbf{n}[\{x: f(x)\geq s\}, b] = u \quad \text{and} \quad \mathbf{n}[G, (b_1, \ldots, b_n, s)] = (u_1, \ldots, u_n, 0)$$

whenever $\lambda(b) < s < \mu(b)$.

(18) *If $n>1$, R and σ are as in 4.5.2, $t\in\mathbf{R}$,*

$$\mathscr{L}^n[R\cap\{x: f(x)>t\}] \leq \mathscr{L}^n(R)/2, \quad \mathscr{L}^n[R\cap\{x: f(x)<t\}] \leq \mathscr{L}^n(R)/2,$$

and $\beta = n/(n-1)$, then

$$\left(\int_R |f(x)-t|^\beta\, d\mathscr{L}^n x\right)^{1/\beta} \leq \sigma\, \|\partial T\|(R).$$

(19) *If $n>1$, σ is as in 4.5.3, $b\in\mathbf{R}^n$, $0<\rho<\infty$, $t\in\mathbf{R}$,*

$$\mathscr{L}^n[\mathbf{U}(b,\rho)\cap\{x: f(x)>t\}] \leq \alpha(n)\,\rho^n/2,$$

$$\mathscr{L}^n[\mathbf{U}(b,\rho)\cap\{x: f(x)<t\}] \leq \alpha(n)\,\rho^n/2,$$

and $\beta = n/(n-1)$, then

$$\left(\rho^{-n} \int_{\mathbf{U}(b,\rho)} |f(x)-t|^\beta\, d\mathscr{L}^n x\right)^{1/\beta} \leq \sigma\, \rho^{1-n}\, \|\partial T\|\, \mathbf{U}(b,\rho).$$

(20) *If $\lambda(b)=\mu(b)\in\mathbf{R}$, $n>1$, $\beta=n/(n-1)$ and σ is as in 4.5.3, then*

$$\limsup_{\rho\to 0+} \left(\rho^{-n} \int_{\mathbf{U}(b,\rho)} |f(x)-F(b)|^\beta\, d\mathscr{L}^n x\right)^{1/\beta} \leq \sigma\, \alpha(n-1)\, \Theta^{*n-1}(\|\partial T\|, b).$$

(21) *If $n>1$, then, for $\mathscr{H}^{n-1}$ almost all b in $\mathbf{R}^n \sim E$,*

$$\lim_{\rho\to 0+} \rho^{-n} \int_{\mathbf{U}(b,\rho)} |f(x)-F(b)|^{n/(n-1)}\, d\mathscr{L}^n x = 0.$$

(22) *If $n>1$, then for $\mathscr{H}^{n-1}$ almost all b in E the vector u characterized by (17) satisfies also the conditions*

$$\lim_{\rho\to 0+} \rho^{-n} \int_{Q^+(b,\rho)} |f(x)-\lambda(b)|^{n/(n-1)} \, d\mathscr{L}^n x = 0$$

with $Q^+(b,\rho) = \mathbf{U}(b,\rho) \cap \{x: (x-b)\bullet u > 0\}$, *and*

$$\lim_{\rho\to 0+} \rho^{-n} \int_{Q^-(b,\rho)} |f(x)-\mu(b)|^{n/(n-1)} \, d\mathscr{L}^n x = 0$$

with $Q^-(b,\rho) = \mathbf{U}(b,\rho) \cap \{x: (x-b)\bullet u < 0\}$.

(23) *If $n=1$, U is an open interval and*

$$r = \inf\{s: \mathscr{L}^1[U \cap \{x: f(x) < s\}] > 0\},$$

$$t = \sup\{s: \mathscr{L}^1[U \cap \{x: f(x) > s\}] > 0\},$$

then $t - r \le \|\partial T\|(U)$; *consequently*

$$\mathbf{V}_a^b F \le \|\partial T\| \{x: a \le x \le b\} \quad \text{for} \quad -\infty < a < b < \infty,$$

$$\{\lambda(b), \mu(b)\} = \{\lim_{x\to b-} F(x), \lim_{x\to b+} F(x)\} \quad \text{for} \quad b \in \mathbf{R}.$$

(24) *For $\mathscr{H}^{n-1}$ almost all b in $\mathbf{R}^n$,*

$$F(b) = \lim_{\varepsilon\to 0+} \int f(b+\varepsilon z)\, \psi(|z|)\, d\mathscr{L}^n z = \lim_{\varepsilon\to 0+} \int f(x)\, \varepsilon^{-n} \psi(\varepsilon^{-1}|b-x|)\, d\mathscr{L}^n x$$

whenever ψ is a real valued $\mathscr{L}^1$ measurable function with compact support such that

$$\mathscr{H}^{n-1}(\mathbf{S}^{n-1}) \int_0^\infty r^{n-1} \psi(r)\, d\mathscr{L}^1 r = 1 \quad \text{and} \quad \int_0^\infty r^{n-1} |\psi(r)|^n \, d\mathscr{L}^1 r < \infty;$$

furthermore $\langle T, \mathbf{1}_{\mathbf{R}^n}, b\rangle = F(b)\, \delta_b$.

(25) *For $\|\partial T\|$ almost all b in $\mathbf{R}^n \sim E$,*

$$\overrightarrow{\partial T}(b) = * \mathbf{n}[\{x: f(x) \ge F(b)\}, b] = -(\wedge_{n-1} p)\, \eta/|\eta|$$

where $\eta = \vec{S}(b_1, \ldots, b_n, F(b)) \llcorner Y_{n+1}$.

(26) *For $\mathscr{L}^n$ almost all b the function f has an $\mathscr{L}^n$ approximate differential L at b such that*

(I) *either* $\Theta^n(\|\partial T\|, b) = 0$ *and* $L = 0$,
or $L = -\mathbf{D}_{n-1}[\Theta^n(\|\partial T\|, b)\, \overrightarrow{\partial T}(b)]$;

(II) *in case $n>1$ and $\beta = n/(n-1)$,*

$$\lim_{\rho\to 0+} \rho^{-n} \int_{\mathbf{U}(b,\rho)} \left| \frac{f(x)-f(b)-L(x-b)}{|x-b|} \right|^\beta d\mathscr{L}^n x = 0;$$

(III) *in case $n=1$, L is the differential of F at b;*

(IV) $\vec{S}(b_1, \ldots, b_n, f(b)) = (-1)^n\, \mathbf{D}^1(M/|M|)$ *with*

$$M = Y_{n+1} - (L \circ p) \in \wedge^1 \mathbf{R}^{n+1}.$$

(27) *If* $n>1$, $i\in\{1,2,\ldots,n\}$,

$$\Omega_i=(-1)^i DX_1\wedge\cdots\wedge DX_{i-1}\wedge DX_{i+1}\wedge\cdots\wedge DX_n,$$

$$q_i(x)=(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)\in\mathbf{R}^{n-1} \text{ for } x\in\mathbf{R}^n,$$

$$p_i(y)=(y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_{n+1})\in\mathbf{R}^n \text{ for } y\in\mathbf{R}^{n+1},$$

$$\chi_{i,z}(t)=(z_1,\ldots,z_{i-1},t,z_i,\ldots,z_{n-1})\in\mathbf{R}^n \text{ for } z\in\mathbf{R}^{n-1},\ t\in\mathbf{R},$$

and if W is a Borel subset of $\mathbf{R}^n$, *Z is a Borel subset of* $\mathbf{R}^{n-1}$, $-\infty<\alpha<\beta<\infty$, $\gamma\in\mathscr{D}^0(\mathbf{R}^n)$, $\operatorname{spt}\gamma\subset\{x:\alpha<x_i<\beta\}$, *then*

$$\|(\partial T)\llcorner\Omega_i\|(W)=\int N[p_i|C\cap p^{-1}(W),v]\,d\mathscr{L}^n v,$$

$$\|(\partial T)\llcorner\Omega_i\|\{x: q_i(x)\in Z, \alpha<x_i<\beta\}=\int_Z \lim_{\delta\to 0+}\mathbf{V}_{\alpha+\delta}^{\beta-\delta}(F\circ\chi_{i,z})\,d\mathscr{L}^{n-1}z,$$

$$(\partial T)(\gamma\wedge\Omega_i)=\int\int_\alpha^\beta(\gamma\circ\chi_{i,z})\,d(F\circ\chi_{i,z})\,d\mathscr{L}^{n-1}z.$$

(28) *If* $n=1$, *W is a Borel subset of* $\mathbf{R}$, $-\infty<\alpha<\beta<\infty$, $\gamma\in\mathscr{D}^0(\mathbf{R})$, $\operatorname{spt}\gamma\subset\{x:\alpha<x<\beta\}$, *then*

$$\|\partial T\|(W)=\int N[Y_2|C\cap p^{-1}(W),v]\,d\mathscr{L}^1 v,$$

$$\|\partial T\|\{x:\alpha<x<\beta\}=\lim_{\delta\to 0+}\mathbf{V}_{\alpha+\delta}^{\beta-\delta}F,\quad (\partial T)(-\gamma)=\int_\alpha^\beta\gamma\,dF.$$

(29) *In case $n>1$ the following three conditions are equivalent:*

(I) *For* $W\subset\mathbf{R}^n$, $\mathscr{H}^{n-1}(W)<\infty$ *implies* $\|\partial T\|(W)=0$.

(II) *F is* ($\mathscr{L}^n$) *approximately continuous at* $\mathscr{H}^{n-1}$ *almost all points of* $\mathbf{R}^n$.

(III) *For* $i=1,2,\ldots,n$ *the functions* $F\circ\chi_{i,z}$ *corresponding to* $\mathscr{L}^{n-1}$ *almost all z in* $\mathbf{R}^{n-1}$ *are continuous on* $\mathbf{R}$.

In case $n=1$ the equivalence holds provided (III) *is replaced by the condition that F be a continuous function.*

(30) *In case $n>1$ the following six conditions are equivalent:*

(I) *For* $W\subset\mathbf{R}^n$, $\mathscr{L}^n(W)=0$ *implies* $\|\partial T\|(W)=0$.

(II) *For* $W\subset\mathbf{R}^n$, $\mathscr{L}^n(W)=0$ *implies* $\mathscr{H}^n[C\cap p^{-1}(W)]=0$.

(III) $(\partial T)\llcorner\Omega_i=\mathscr{L}^n\llcorner D_i F$ *for* $i=1,2,\ldots,n$.

(IV) $\|(\partial T)\llcorner\Omega_i\|=\mathscr{L}^n\llcorner|D_i F|$ *for* $i=1,2,\ldots,n$.

(V) *For* $i=1,2,\ldots,n$ *the functions* $F\circ\chi_{i,z}$ *corresponding to* $\mathscr{L}^{n-1}$ *almost all z in* $\mathbf{R}^{n-1}$ *are absolutely continuous on* $\mathbf{R}$.

(VI) *There exists a sequence of functions* $f_j\in\mathscr{E}^0(\mathbf{R}^n)$ *such that, for every compact* $K\subset\mathbf{R}^n$,

$$\lim_{j\to\infty}\int_K|f_j-f|\,d\mathscr{L}^n=0 \quad\text{and}\quad \lim_{(j,k)\to(\infty,\infty)}\int_K\|Df_j-Df_k\|\,d\mathscr{L}^n=0.$$

Furthermore (VI) *implies, for every compact* $K \subset \mathbf{R}^n$,

$$\lim_{j\to\infty} \int_K \|D f_j - \operatorname{ap} D f\| \, d\mathscr{L}^n = 0 .$$

In case $n=1$ *the equivalence holds provided* (V) *is replaced by the condition that* F *be an absolutely continuous function, and* Ω_1 *is replaced by* -1.

(31) *If* $n>1$, $\beta = n/(n-1)$ *and* $\mathbf{M}(\partial T) < \infty$, *then there exists a unique* $c \in \mathbf{R}$ *such that*

$$\left[\int |f(x)-c|^\beta \, d\mathscr{L}^n x\right]^{1/\beta} \le n^{-1} \alpha(n)^{-1/n} \mathbf{M}(\partial T).$$

In case $\operatorname{spt} f$ *is compact, then* $c=0$.

In case f *is integer valued, then* c *is an integer and*

$$\mathbf{M}(T - c\,\mathbf{E}^n)^{1/\beta} \le n^{-1} \alpha(n)^{-1/n} \mathbf{M}(\partial T).$$

Proof of (1). In this case we define the locally Lipschitzian homeomorphism h of $\mathbf{R}^{n+1}$ onto $\mathbf{R}^{n+1}$ by the formula

$$h(y) = (y_1, \ldots, y_n, y_{n+1} + f(y_1, \ldots, y_n)) \text{ for } y \in \mathbf{R}^{n+1},$$

observe that $G = h\{y\colon y_{n+1} \le 0\}$, $g = h \circ p^*$, and use 4.1.26, 4.1.8 to compute

$$\begin{aligned} S &= (-1)^n \, \partial h_\# (\mathbf{E}^{n+1} \llcorner \{y\colon y_{n+1} \le 0\}) \\ &= h_\# [(-1)^n \, \partial(\mathbf{E}^{n+1} \llcorner \{y\colon y_{n+1} \le 0\})] = h_\# \, p^*_\# \, \mathbf{E}^n = g_\# \, \mathbf{E}^n, \end{aligned}$$

because $\partial[\mathbf{E}^n \times (\mathbf{E}^1 \llcorner \{t\colon t \le 0\})] = (-1)^n \mathbf{E}^n \times \delta_0$. From 4.1.25 and 3.2.3 we see that

$$\|S\| = \mathscr{H}^n \llcorner \operatorname{im} g = g_\# (\mathscr{L}^n \llcorner J_n g),$$

since g is univalent, and for $\mathscr{L}^n$ almost all x we obtain

$$D_i g(x) = \varepsilon_i + D_i f(x)\, \varepsilon_{n+1} \text{ for } i = 1, \ldots, n,$$

$$\begin{aligned} \langle e_1 \wedge \cdots \wedge e_n, \textstyle\bigwedge_n D g(x) \rangle &= D_1 g(x) \wedge \cdots \wedge D_n g(x) \\ &= \varepsilon_1 \wedge \cdots \wedge \varepsilon_n + \sum_{i=1}^n D_i f(x)\, \varepsilon_1 \wedge \cdots \wedge \varepsilon_{i-1} \wedge \varepsilon_{n+1} \wedge \varepsilon_{i+1} \wedge \cdots \wedge \varepsilon_n , \end{aligned}$$

$$[J_n g(x)]^2 = 1 + \sum_{i=1}^n [D_i f(x)]^2 = 1 + |D f(x)|^2 .$$

Thus we have verified (1). However we proceed to amplify our discussion of the locally Lipschitzian case by some observations which will be useful in the sequel. Noting that $p \circ g = \mathbf{1}_{\mathbf{R}^n}$ we obtain

$$p_\# \|S\| = \mathscr{L}^n \llcorner (1 + \|D f\|^2)^{\frac{1}{2}} \le \mathscr{L}^n \llcorner (1 + \|D f\|) = \mathscr{L}^n + \|\partial T\| .$$