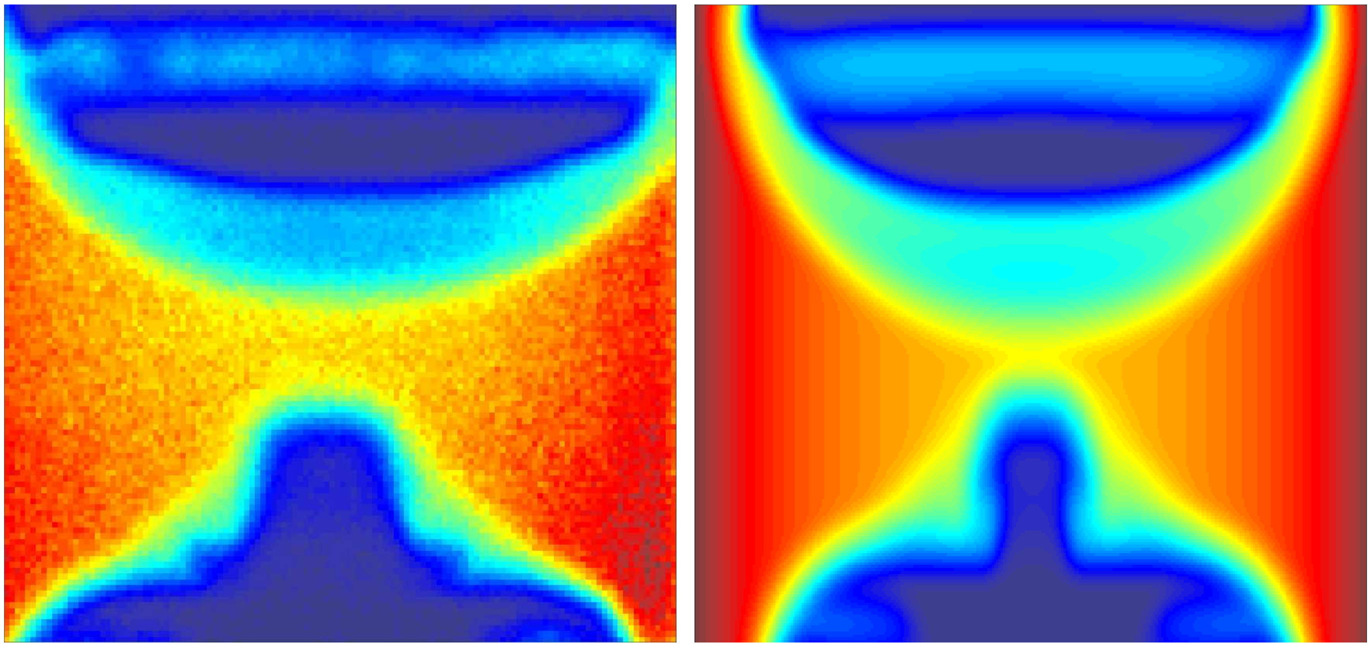
Geometric Measures I: Symmetric Rearrangements

Given images we can ask how to use the geometry represented by the image to generate measure. That can be compared robustly. In particular we are interested in validation of simulation codes. What is the distance between the following experiment and simulation?



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Idea: generate a rigid transformation invariant signature by looking at signatures derived from the *Steiner symmetrizations* or *symmetric decreasing rearangements*.

Symmetric decreasing rearrangements: for f in $L^p(\mathbb{R}^n)$, the *symmetric decreasing rearrangement* f^* is the $L^p(\mathbb{R}^n)$ function such that $\{f^* \geq y\}$ is a disk centered at the origin in $\mathbb{R}^n$ such that $\mathcal{H}^n(\{f^* \geq y\}) = \mathcal{H}^n(\{f \geq y\})$. We will denote the mapping from f to f^* by $\mathcal{R}$.

$||f||_p = ||f^*||_p$: This is a consequence of the fact that the L^p norms are integrals over areas of level sets.

$||\nabla f||_p \geq ||\nabla f^*||_p$: used in applications to variational problems.

$\mathcal{R}$ is *not* continuous in $W^{1,p}$! We will return to this and other theoretical details later.

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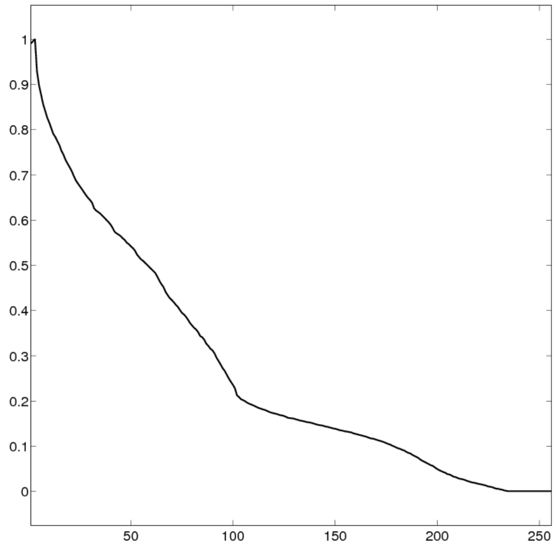
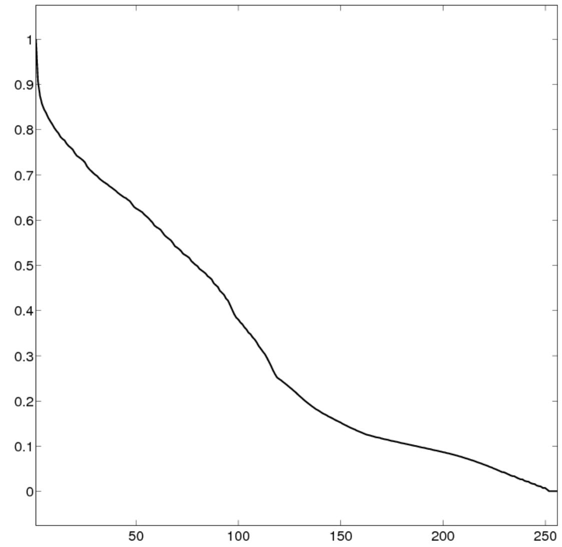
Easiest implementation of idea to use symmetric decreasing rearangements: use the areas of the disks as a function of height of the disk (f^*) as a signature.





Now compare these with a simple 1-D warp.

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How do we use these area signatures (symmetric rearrangements)?

Regularize: run *mean curvature flow* a bit on the simulated images and the experimental images.

Compute area signatures: compute areas of the level sets.

1-D Warping: use a simple warping method to compare the 1-D signatures.

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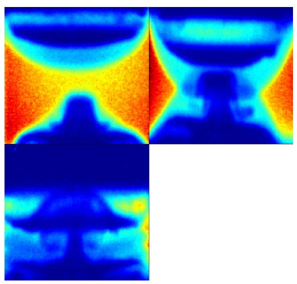
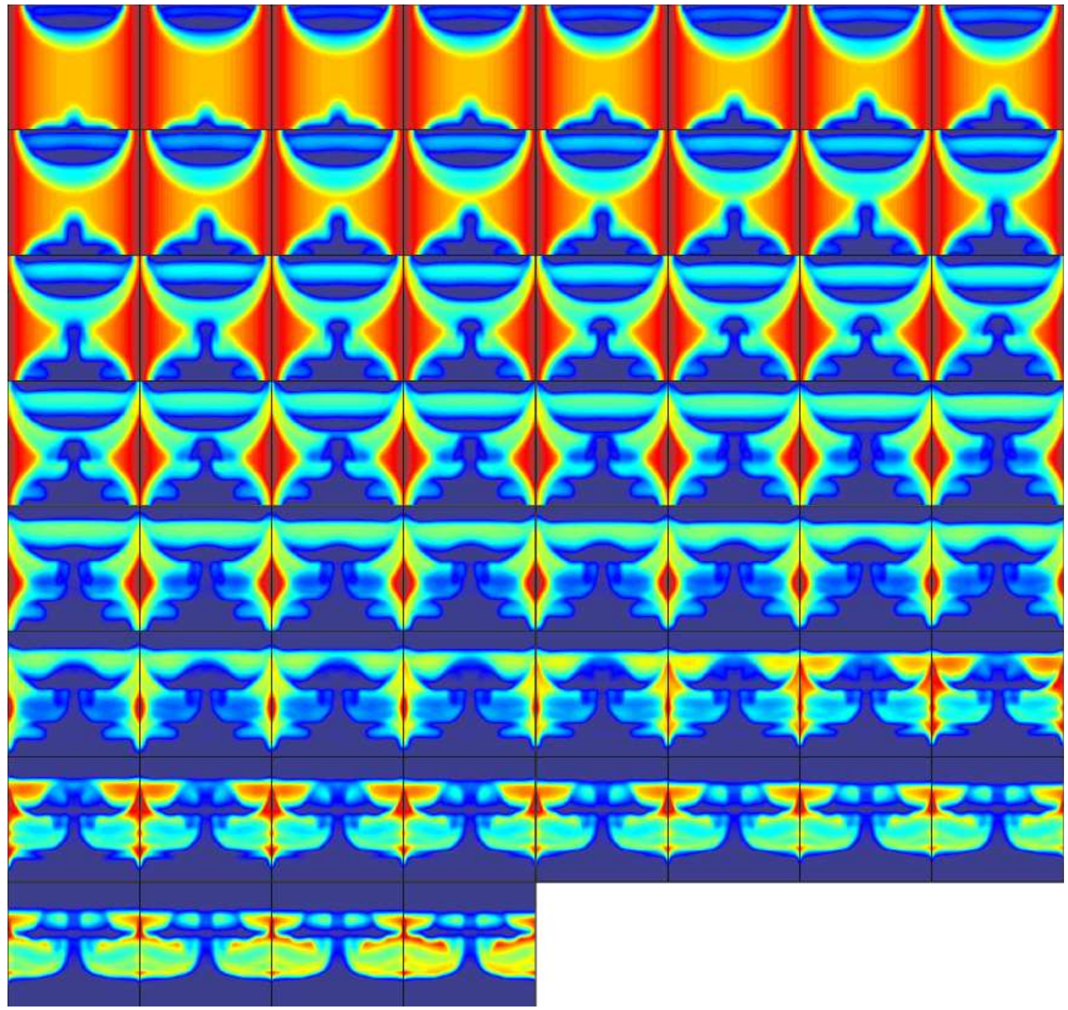
The experimental and simulated images.





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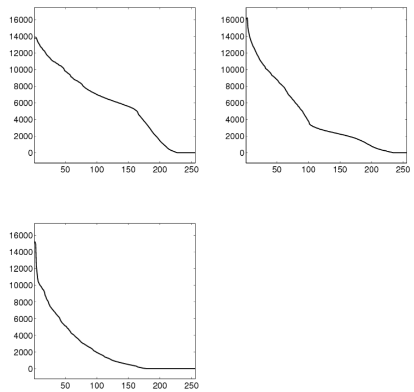
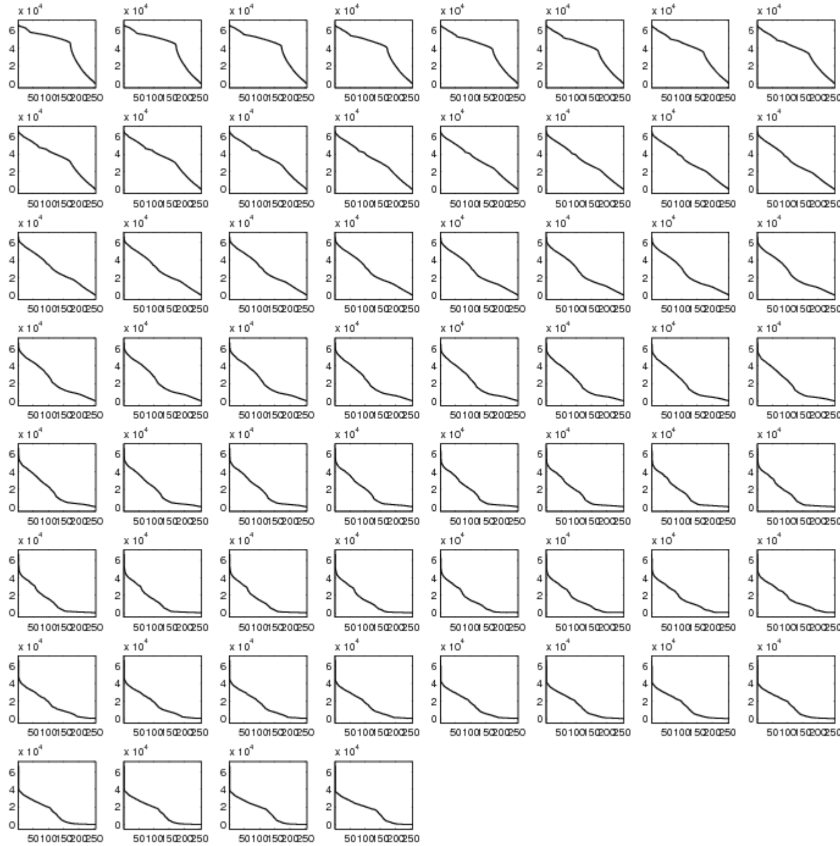
The results: area signatures





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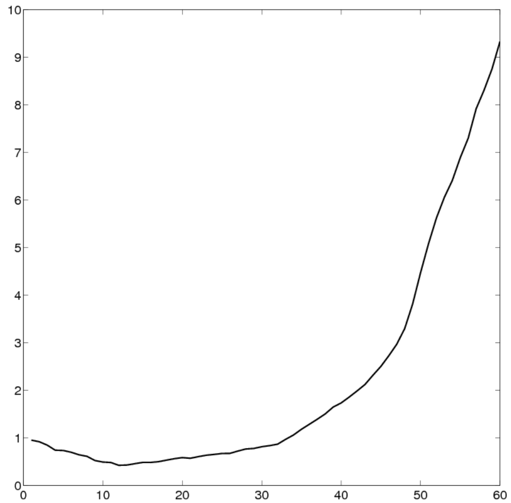
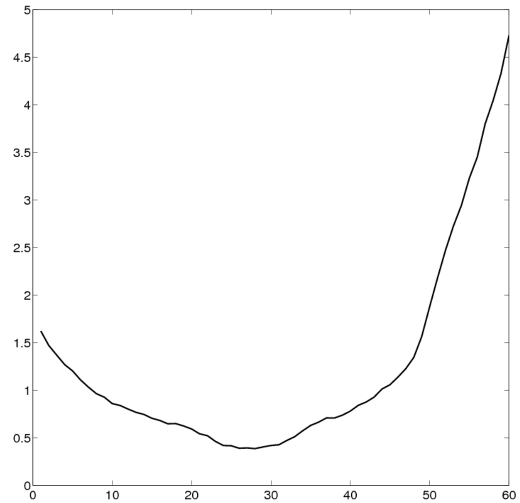
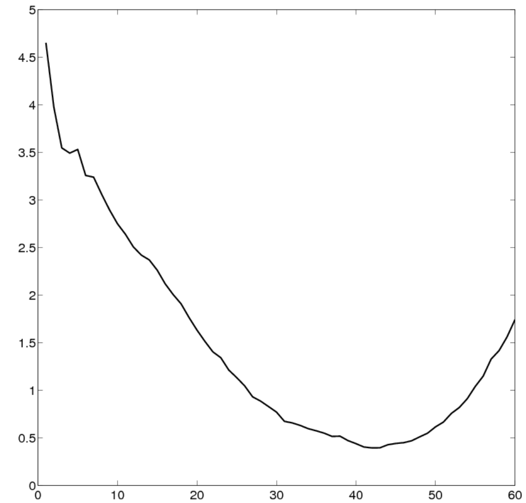
The reuslts: the registration problem is in fact a very good test of quality of the metric – validated by expert judgement.







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Back to theoretical questions.

Fred Almgren and Elliott Lieb showed (in 1989) the rather surprising result that $\mathcal{R}$ is discontinuous in $W^{1,p}$. Should this cause us to wonder about stability of our results?

Answer: no

The points of $W^{1,p}$ for which $\mathcal{R}$ misbehaves are exactly the *coarea irregular functions*.

Bad news: *coarea irregular functions* are dense in $W^{1,p}$
Not so bad news: *coarea regular functions* are also dense in $W^{1,p}$
Even better news: Coarea irregular functions are rather strange and contrived.

Moral: Don't worry, be happy: symmetric rearrangements will work for you!

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So what the heck are coarea regular and coarea irregular functions?

Coarea regular Define $\mathcal{G}_f(y) = \int \chi_{\{f>y\}}\chi_{\{\nabla f=0\}} d\mathcal{L}^n$. If $\frac{d}{dy}\mathcal{G}_f(y)$ is singular wrt $\mathcal{L}^1$ on $\mathbb{R}$, it is *coarea regular*.

Coarea irregular: not coarea regular – $\mathcal{L}^n(\{\nabla f = 0\}) > 0$ and there are no open sets in $\{\nabla f = 0\}$

A stab at understanding: the details can be found in a very technical 1989 paper by Fred Almgren and Elliott Lieb, but a key point (I think) is that any coarea irregular function f can be seen to be arbitrarily close (in L^p) to a function g such that $\mathcal{L}^n(\{\nabla f = 0\}) > \mathcal{L}^n(\{\nabla g = 0\}) + \delta$, $\delta > 0$ and indpendent of g.

Tidbits: All functions in $W^{1,p}$ on $\mathbb{R}^1$ are coarea regular: hence Coron's 1984 proof of the continuity of $\mathcal{R}$ in $\mathbb{R}^1$. There exist $C^{n-1,\gamma}$ coarea irregular functions in $\mathbb{R}^n$ as long as $\gamma < 1$. $C^{n-1,1}$ implies coarea regularity.