Geometry from Deriviatives: from edges to tubes

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1 Introduction

Interactions between analysis and geometry are rich and pervasive: it is hard to imagine one without the other. This is especially true for me given my personal bent towards thinking geometrically about analysis (and everything else).

Motivating question: *How can we use derivatives to extract geometric features and properties from sets?*

We will primarily be interested in sets which are subsets of 2 and 3-dimensional space. Figure 1 illustrates the extraction of an edge and its properties from an initial set or shape

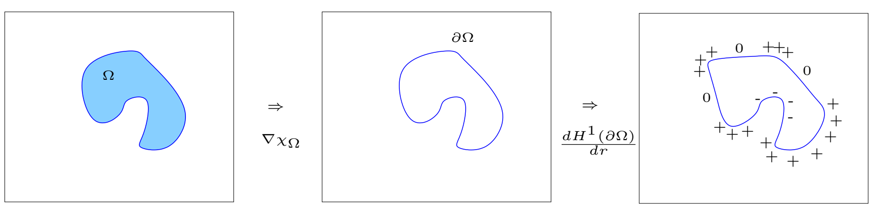

Figure 1: Schematic illustrating the extraction of geometric information from sets in Euclidean space.

Ω. The details of the derivatives used ($\nabla\chi_\Omega$ and $\frac{d\mathcal{H}^1(\partial\Omega)}{dr}$) will be explained later in this talk. The same tasks can be attempted for objects in metric spaces. At the end of this talk[1], I will add a little for the experts on how to do this in metric spaces.

For most people, analysis starts with a course in calculus. Some of these courses are rigorous, using ϵ's and δ's and proofs, while most are non-rigorous and focused on computation and some intuitive explanations. Since many in this audience might be a little rusty with their calculus, I start with a review of the pieces of calculus needed for this talk.

[1]This paper is based on the talk I gave at the conference to honor my undergraduate mentors Ken Wiggins and Tom Thompson. It was held at Walla Walla University on May 17, 2013. For more information, see the conference webpage here http://math.wallawalla.edu/conferences/TomAndKen/

2 A Very Fast Review of Calculus

Derivatives: the classic definition: When you took calculus possibly a long time ago, you were taught that the derivative $\frac{df}{dx}(x)$ was simply the slope of f at x after you zoomed in on it. To be more precise, it was the limit of the slopes of the secant lines from x to $x+\epsilon$ as $\epsilon \to 0$:

$$\frac{df}{dx} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon} = \text{slope of the tangent line} \tag{1}$$

Because this is the slope of the line that the graph of f converges to as we zoom into the graph – the tangent line of f at x – we say that the derivative is the slope of the tangent line to f at x.

We will use the fact that the change in f, which we denote by Δf, that is caused by changing x by an amount Δx is given (approximately) by:

$$\Delta f \approx \frac{df}{dx} \Delta x. \tag{2}$$

The approximation is good long as Δx is small[2]. All of this is illustrated in Figure 2.

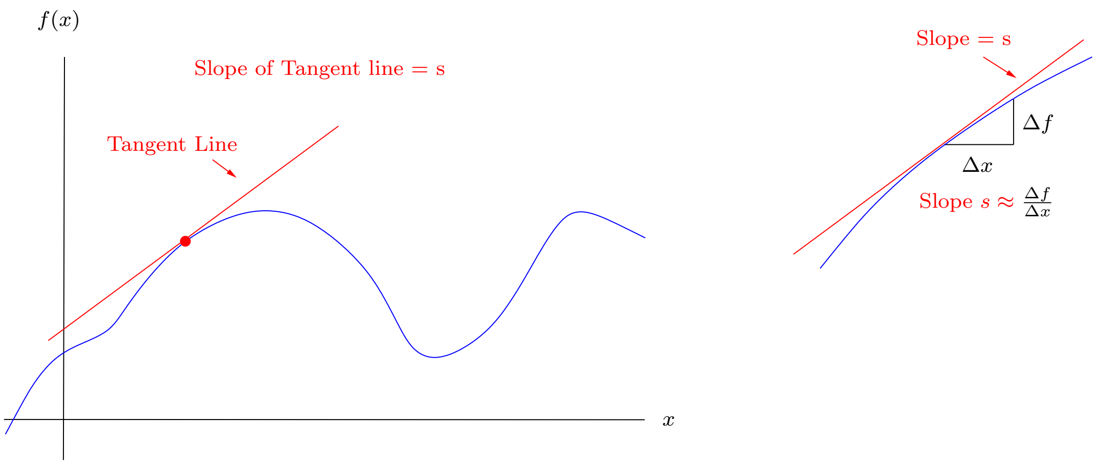

Figure 2: The classic derivative is the slope of the tangent line.

[2]Roughly, small means Δx is small enough that on that scale, the graph looks like a straight line.

What does $\int_a^b f(x)dx$ mean? Very simply put the integral of the function f from $x = a$ to $x = b$, $\int_a^b f(x)dx$, is the area under the graph of f from a to b. See figure 3.

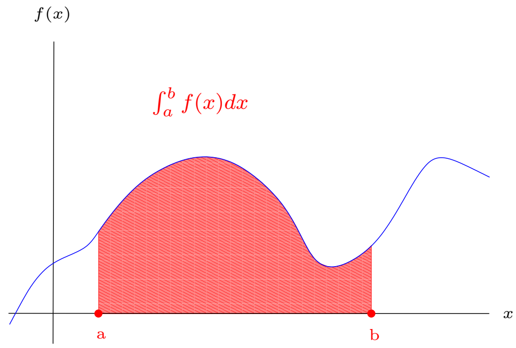

Figure 3: The integral of f

How do we calculate the integral? We can compute the integral by approximating the region under the graph by a many narrow rectangles. As we let the thickness of the rectangles go to zero, the error in the approximation goes to zero. (There are better definitions, but this one will suffice for our purposes.) This method of calculating the integral is illustrated in Figure 4:

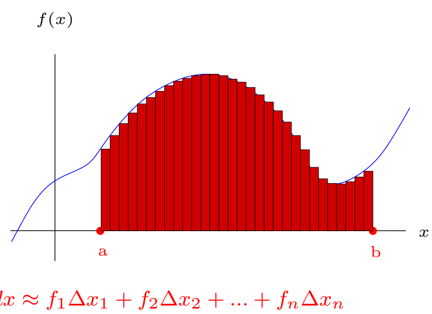

Figure 4: In this figure, $\Delta x_1 = \Delta x_2 = ... = \Delta x_n$. The smaller the Δx's (i.e. the skinnier the rectangles) the better the approximation. In the limit as all the Δx's go to zero, we get the right answer.

The Fundamental Theorem: Looking at what happens when we integrate a derivative leads us to the *Fundamental Theorem of Calculus (FTC)*.

Using what we have developed so far and abbreviating $\frac{df}{dx}$ with f', we get:

$$\begin{aligned}\int_a^b f'(x)\,dx &\approx f_1'\Delta x_1 + f_2'\Delta x_2 + ... + f_n'\Delta x_n \\ &\approx \Delta f_1 + \Delta f_2 + ... + \Delta f_n \\ &= f(b) - f(a)\end{aligned}$$

$$FTC: \quad \int_a^b f'(x)dx = f(b) - f(a)$$

Derivatives of Products: If we plot the size of f(x) as the vertical coordinate and the value of g(x) as the horizontal coordinate, then the value of the product f(x)g(x) is the area of the rectangle with corners, (0,0), (g(x),0), (0,f(x)), and (g(x),f(x)). Using the visualization, we can easily deduce what the derivative of the product is.

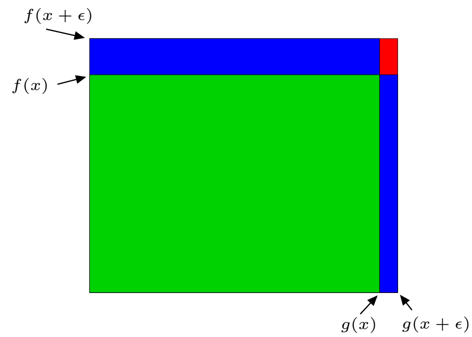

Figure 5: The product of two functions can be viewed schematically as a two dimensional area.

Using: $f(x+\epsilon) = f(x) + f'(x)\epsilon$ (a harmless fiction)

$$\begin{aligned} (fg)' &= \lim_{\epsilon\to 0} \frac{f(x+\epsilon)g(x+\epsilon) - f(x)g(x)}{\epsilon} \\ &= \lim_{\epsilon\to 0} \left(f'(x)g(x) + f(x)g'(x) + f'(x)g'(x)\epsilon \right) \\ &= f'(x)g(x) + f(x)g'(x) \text{ (the pure, unadulterated truth!)} \end{aligned}$$

3 Extracting Geometric Features

3.1 Generalizing the Derivative

Motivation for a new definition: Using derivatives to find places where a function is steep in slope is of course very natural, but when the function we are taking the derivative of is the characteristic function of a set Ω, then the derivative is zero everywhere except on the boundary $\partial\Omega$ where it is undefined (or infinite, depending on your perspective. See Figure 6.

The trick: We now use classic trick – find a definition that is equivalent in the nice case and more widely applicable: we use a definition that is equivalent to the old definition when the functions are nice, but is applicable to some functions that the old definition says lives in the class of functions without derivatives. This is illustrated in figure 7.

The new definition, in steps: The classic definition says that the derivative of f at x is the slope of the tangent line of f at the point $(x, f(x))$:

$$\frac{df}{dx} = \lim_{\epsilon\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Our new approach looks at a property that the derivative has when it is integrated against other functions:

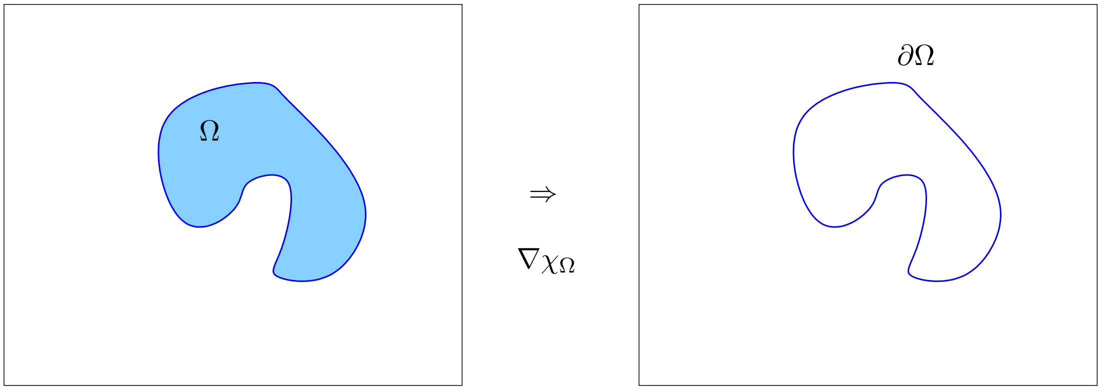

Figure 6: *One can define the edges to be the points where the gradient does not exist. But using the classical gradient is problematic since there is information (for example, the height and direction of the edge) that is lost when the derivative is used to find edges.*

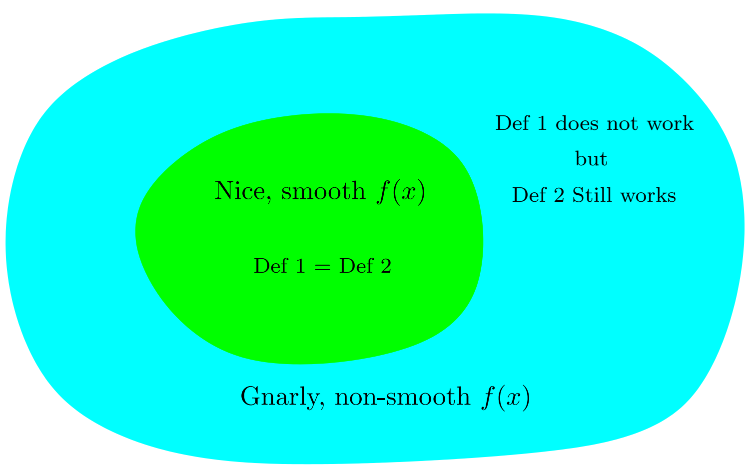

Figure 7: *The Classic path to generalizations in mathematics, schematically.*

Step 1 Start by picking an f; choose an interval [a,b] and set of smooth test functions Φ, $\phi(a) = \phi(b) = 0$ for $\phi \in \Phi$

Step 2 Note that if f is smooth, (using the FTC above)

$$\begin{aligned} 0 &= \phi(b)f(b) - \phi(a)f(a) \\ &= \int_a^b (\phi f)' dx \\ &= \int_a^b \phi f' dx + \int_a^b \phi' f dx \end{aligned}$$

Step 3 Observe that in $\int_a^b \phi f' dx = -\int_a^b \phi' f dx$ everything is defined for very rough functions f except the f'!

New Definition, Version 1: $f' \equiv g$ if $\int_a^b \phi g dx = -\int_a^b \phi' f dx$ for all ϕ.

But we actually want to generalize this a bit more. We do this motivated by an example:

Example of the derivative of a step function Studying the case of a step function (see Figure 8), we find a very useful generalization. Considering the example and version 1 of

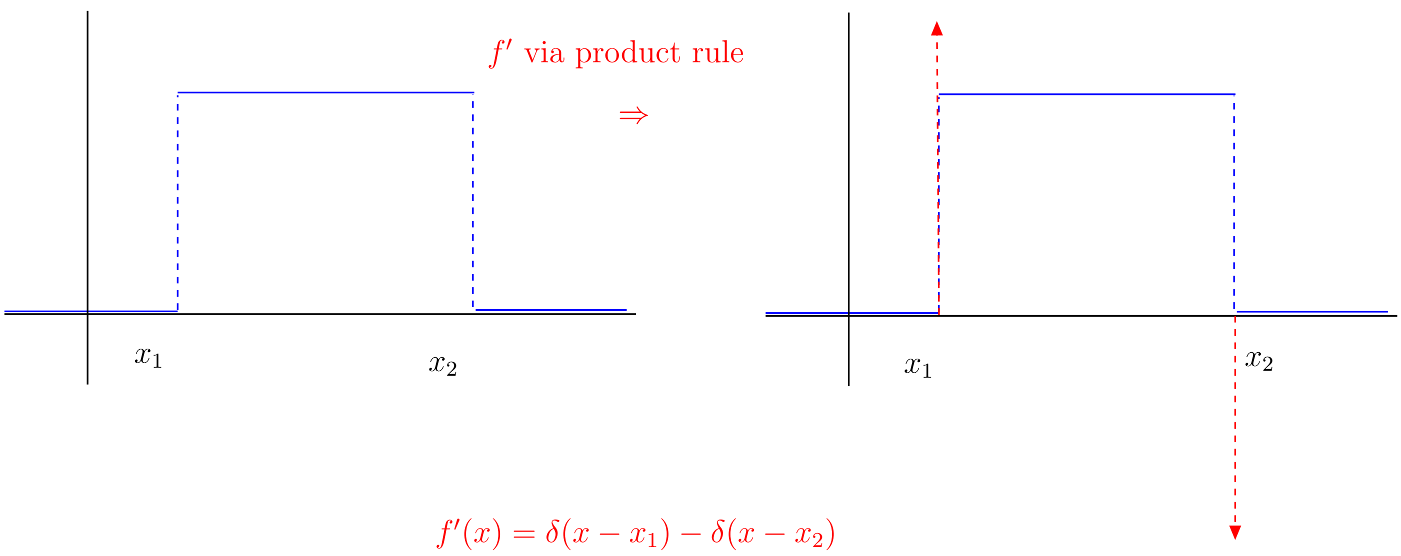

Figure 8: *Attempting to take the derivative of a step function, we see that there is no function that works, even with our new definition. Instead, $f(x)$ is the sum of two signed point mass measures.*

the new definition, we see that if we allow ourselves to pick a function **and** a new measure on the left hand side, we can make it work. That is:

New Definition, version 2 $f' = g d\mu$ if $\int_a^b \phi g d\mu = -\int_a^b \phi' f dx$ for all ϕ.

Jumping up a dimension (or two or three ...): Using the Divergence theorem (a higher dimensional FTC!):

$$\begin{aligned} 0 = \int_{\partial E} (\vec{\phi} f) \cdot \vec{n} d\sigma &= \int_E \nabla \cdot (\vec{\phi} f) dx \\ &= \int_E \vec{\phi} \cdot \nabla f dx + \int_E (\nabla \cdot \vec{\phi}) f dx \end{aligned}$$

leads to something true for smooth f:

$$\int_E \vec{\phi} \cdot \nabla f dx = -\int_E (\nabla \cdot \vec{\phi}) f dx$$

which we now use to find derivatives when f isn't smooth ...

New Definition, final version $f' = \vec{g} d\mu$ if

$$\int_E \vec{\phi} \cdot \vec{g} d\mu = -\int_E (\nabla \cdot \vec{\phi}) f dx$$

for all ϕ.

3.2 Extracting Geometric Features using Gradients

We are now in a position to extract $\partial\Omega$, the boundary of a shape Ω, by the means of a derivative. Defining χ_Ω to be the function which is one on Ω and zero everywhere else, we use the final version of the new definition of derivative to compute $\nabla\chi_\Omega$, the gradient of χ_Ω.

The Result: $\nabla\chi_\Omega = \vec{\zeta} \llcorner \mu$ where $\vec{\zeta}$ is a unit vector field and μ is a radon measure

The reward for all this work is that we now have a completely descriptive (lossless even) representation of $\partial\Omega$. (See Figure 9)

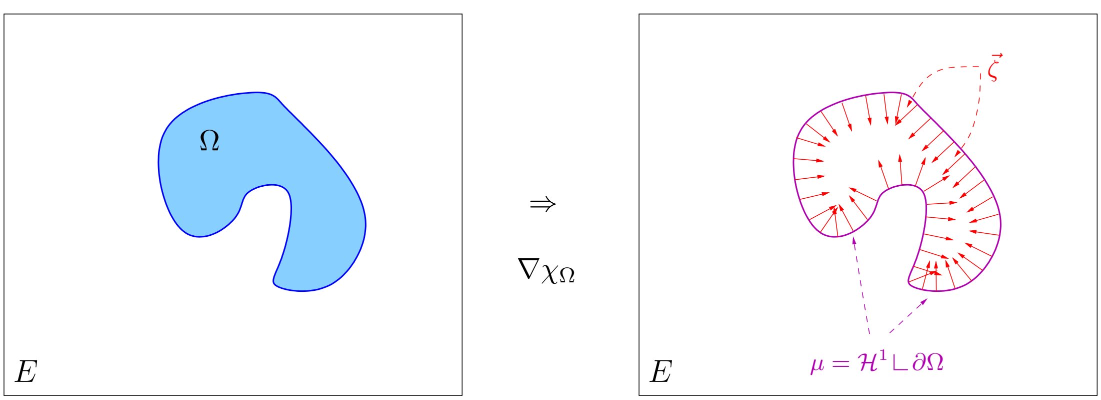

Figure 9: *Using the new definition to calculate the gradient, we get a lossless representation of the boundary.*

3.3 Extracting Geometric Properties of those Features

Now that we have a way to extract the boundary of a shape, we turn our attention to using derivatives to extract quantitative information from the boundary. To do that we will use a *variation* along the normal field to the boundary. (See Figure 10) The basic idea is that

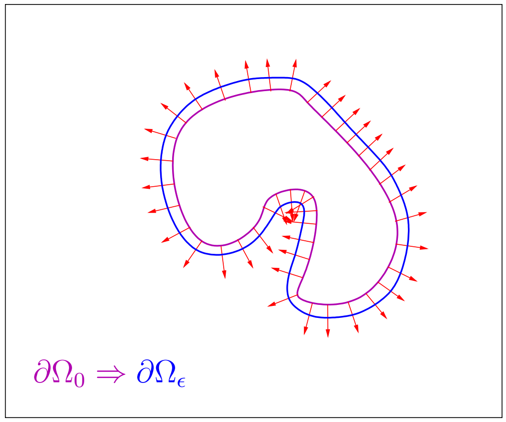

Figure 10: *Flowing the boundary along the normal vector field, the (n-1)-dimensional measure of the boundary changes. As we will see, the exact way in which that (n-1)-measure changes tells us a great deal about the geometry of the boundary.*

the normal to the boundary can be used to create a *normal vector field* when we are close enough to the boundary. How close, close enough is, depends on the curvature. But, if the boundary is smooth, we always have a (possibly small) neighborhood of the boundary in which we can create the normal vector field and take the derivative of the (n-1)-measure of $\partial\Omega$ with respect to the normal field in that neighborhood. As we will see this is enough for our purposes.

Normal Variations: We begin by considering the case in which Ω is 2-dimensional so that $\partial\Omega$ is a 1-dimensional curve. Figure 11 gives a detailed explanation of the fact that the change in length for an infinitesimal piece of the boundary is given by the curvature of the boundary at that point.

The Tube Formula: Using this simple observation about how lengths of curves change as we push out along normal, we can get how the area of a surface changes as we push it out along a normal. We use the fact that any small piece of the surface can be given coordinates so that the the principal curvatures correspond to the two orthogonal directions in the coordinates. Thus the fractional change in area as we push a tiny piece out is the product $(1 + \epsilon\kappa_1)(1 + \epsilon\kappa_2)$. This is illustrated in Figure 12. We can use this to get the change in the area of a boundary as we move out along the normals. We simply integrate

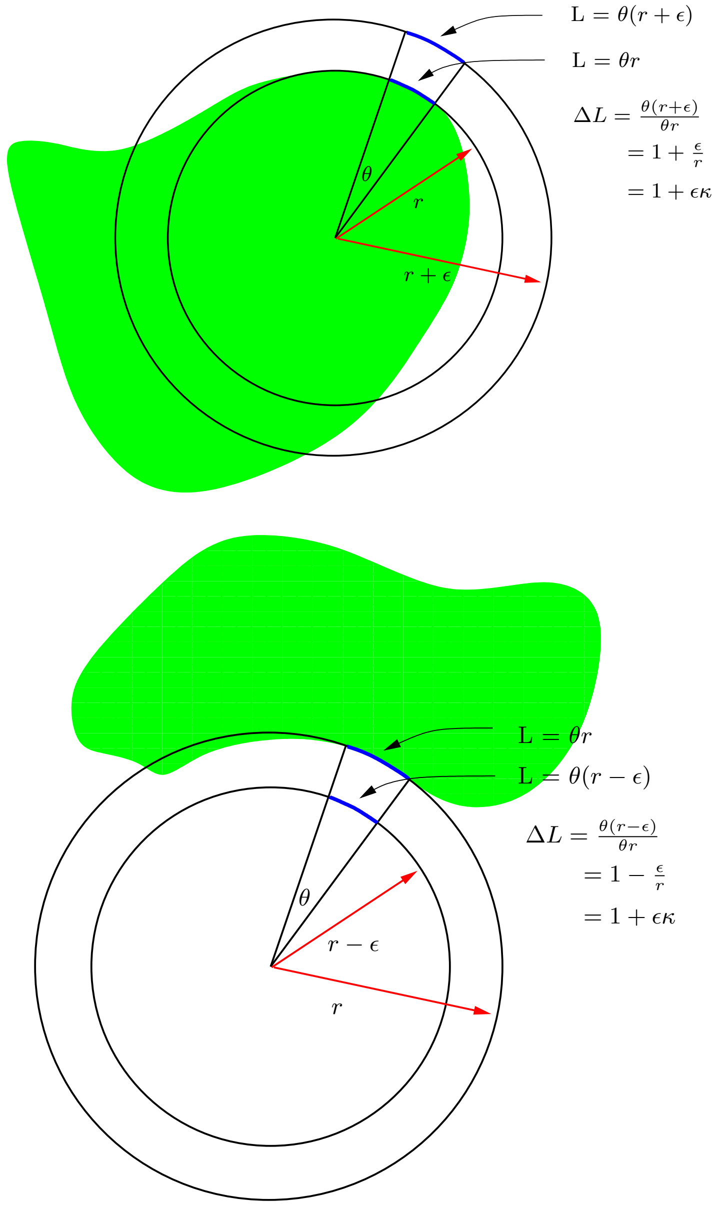

Figure 11: *(top figure) The piece of the boundary expands in length as it would if it were part of a circle with radius equal to $\frac{1}{\kappa}$, where κ is the curvature of the boundary that infinitesimal piece. We get that the factor by which the length grows is equal to $1 + \epsilon\kappa$ where ϵ is the distance we have pushed out along the normals. (bottom figure) Negative curvature yields the same result!*

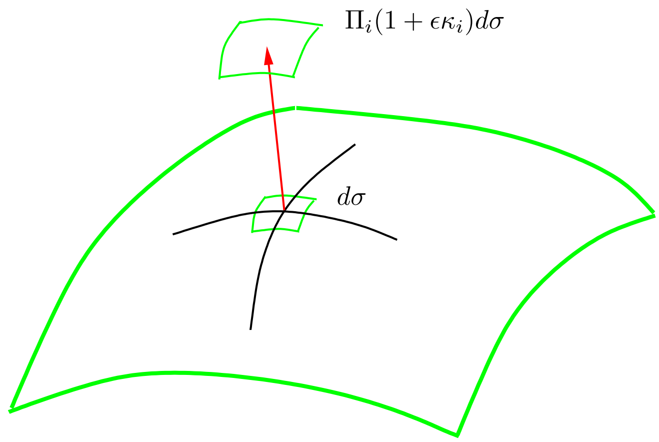

Figure 12: *The change in area of a small piece of $\partial\Omega$ is given by the product of the changes in lengths along the two orthogonal directions.*

this product over the boundary:

$$\begin{aligned}
\mathcal{H}^2(\partial(\Omega &+ B_\epsilon)) \\
&= \int_{\partial\Omega}(1+\epsilon\kappa_1)(1+\epsilon\kappa_2)d\mathcal{H}^2 \\
&= \int_{\partial\Omega} 1 d\mathcal{H}^2 + \epsilon\int_{\partial\Omega}(\kappa_1+\kappa_2)d\mathcal{H}^2 + \epsilon^2\int_{\partial\Omega}\kappa_1\kappa_2 d\mathcal{H}^2 \\
&= \mathcal{H}^2(\partial\Omega) + \epsilon\int_{\partial\Omega}\underbrace{(\kappa_1+\kappa_2)}_{\text{Mean Curvature}} d\mathcal{H}^2 + \epsilon^2\int_{\partial\Omega}\underbrace{\kappa_1\kappa_2}_{\text{Gaussian Curvature}} d\mathcal{H}^2
\end{aligned}$$

Using a curvilinear version of Fubini's Theorem called the coarea formula we get:

$$\begin{aligned}
\mathcal{H}^3(\Omega + B_r) &= \mathcal{H}^3(\Omega) + \int_{\epsilon=0}^{\epsilon=r}\mathcal{H}^2(\partial(\Omega+B_\epsilon))d\epsilon \\
&= \mathcal{H}^3(\Omega) + r\int_{\partial\Omega} 1 d\mathcal{H}^2 + \frac{r^2}{2}\int_{\partial\Omega}(\kappa_1+\kappa_2)d\mathcal{H}^2 \\
&\quad + \frac{r^3}{3}\int_{\partial\Omega}\kappa_1\kappa_2 d\mathcal{H}^2 \\
&= a_0 + a_1 r + a_2 r^2 + a_3 r^3
\end{aligned}$$

where the a_i are integrals of various geometric measures.[3] These geometric measure are: volume of Ω (a_0), area of the boundary $\partial\Omega$ (a_1), integral of the mean curvature of $\partial\Omega$ (a_2), and the integral of Gauss curvature which is a topological constant (a_3).

[3] We have to stay in the *reach* of Ω for this formula to be true. This is the neighborhood mentioned earlier, at the beginning of this section.

4 For the Experts: Can we do any of this in metric spaces?

The basic problem with taking any of this to metric spaces is that metric spaces are not necessarily endowed with an underlying vector space structure. And it is this vector space structure that we have been exploiting in our above work using derivatives.

But all is not lost! Though it is a relatively new area, there has been a lot of work on analysis in metric spaces. And there are ideas that are reminiscent of how we generalized derivatives that we can exploit to get what we want in metric spaces.

I now describe an approach that I am just now exploring with collaborators.

Mean Curvature from Variations: For nice enough $f : \mathbb{R}^n \to \mathbb{R}$:

$$\frac{d}{d\tau} \int |\nabla f + \tau \nabla \phi| dx \bigg|_{\tau=0} = \int \underbrace{\nabla \cdot \frac{\nabla f}{|\nabla f|}}_{\text{Mean Curvature}} \phi \, dx$$

In this case, nice enough means that f is C^1 and $f \neq 0$ on the support of ϕ. The term labeled "mean curvature" is the mean curvature of the level set of f at whatever point that term is evaluated.

Upper Gradients: In metric spaces M we don't have ∇f, but we do have a notion of $|\nabla f|$ in a metric space called *upper gradients*. And that is enough! Very briefly, an upper gradient $g(x)$ of $f(x) : M \to \mathbb{R}$ is a function that has the property that, for almost all rectifiable curves $\gamma : [0, L] \to M$, we have

$$|f(\gamma(L)) - f(\gamma(0))| \leq \int_\gamma g(s) ds$$

where L is the length of γ which is parametrized by arclength and $\gamma([0, L])$ is the image of the curve in the metric space. It is clear that in the case of differentiable functions, ∇f is a minimal upper gradient. (Precisely what is meant by "almost all" curves is a bit technical.)

The idea: Choose a sequence of ϕ_i whose supports are contained in $B(\hat{x}, r_i)$ and $r_i \to 0$ as $i \to \infty$. The idea here is that we will use these test functions to zoom in on some point in the metric space to examine the exact geometric measure of the level set of f there. For "nice enough" f's in metric spaces,

$$\frac{\frac{d}{d\tau} \int |\nabla f + \tau \nabla \phi_i| dx \big|_{\tau=0}}{\int \phi_i dx}$$

should converge to something in a way that is continuous in space. The limit will be interpreted as the mean curvature of the level set of f passing through $\hat{x}$.

Collaborators on this work currently include B. Van Dyke, G. Sandine, and W. Zhu.

5 Notes on References

In this talk we covered a lot of ground. For those whose appetites have been aroused but not satisfied, here are three very good references.

Frank Morgan [4] *Geometric Measure Theory: A beginners guide.* Geometric measure theory is a deeply useful and rather sweeping generalization of differential geometry and other pieces of nonlinear analysis. Frank's book is intended as an introduction to geometric measure theory and was written as an interface to Federer's famous and difficult 1969 tome, *Geometric Measure Theory* [3].

Evans and Gariepy [2] *Measure Theory and fine Properties of Functions.* Evans and Gariepy is my favorite text for advanced analysis and covers some significant pieces of geometric measure theory. In particular, the ideas of generalized derivatives, especially in the spirit of what was presented in the lectures, can be found there. It is where I learned these ideas first.

Bjorn and Bjorn [1] *Nonlinear Potential Theory on Metric Spaces.* This is a very nice reference for the basic ideas of analysis in metric spaces including upper gradients and exactly what is meant by "almost all" rectifiable curves.

References

[1] Anders Björn and Jana Björn. *Nonlinear potential theory on metric spaces*, volume 17. Amer Mathematical Society, 2011.

[2] Lawrence C. Evans and Ronald F. Gariepy. *Measure Theory and Fine Properties of Functions*. Studies in Advanced Mathematics. CRC Press, 1992. ISBN 0-8493-7157-0.

[3] Herbert Federer. *Geometric Measure Theory.-Reprint of the 1969 Edition*. Springer, 1996.

[4] Frank Morgan. *Geometric measure theory: a beginner's guide*. Academic Press, fourth edition, 2008.