Computing $\frac{dF}{du}$ ...

$F(u)$ $\quad u \in C^2(\Omega)$

$\Omega \subset \mathbb{R}^n$

$F : C^2(\Omega) \to \mathbb{R}$

μ : Lebesgue Measure in $\mathbb{R}^n$

$$F(u) = \int_\Omega \nabla u \cdot \nabla u \, d\mu$$

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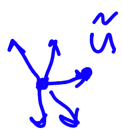
u  $h \equiv \tilde{u} - u$ will be small

$F(u+h) - F(u)$

idea is compute this and find the linear component and "hope" the rest is $o(h)$

$$F(u+h) - F(u) = \underbrace{Ah}_{\text{linear part}} + \overbrace{g(h)}^{o(h)}$$

Compute $\quad F(\underbrace{u+h}) - F(u)$

$h \to \alpha g \quad$ typically choose g $\ni |g| = 1$, $g = 0$ on $\partial\Omega$

$$\int_\Omega (\nabla u + \alpha \nabla g)\cdot(\nabla u + \alpha \nabla g)\, dx \;-\; \int_\Omega \nabla u\cdot \nabla u\, dx$$

$$= \cancel{\int_\Omega \nabla u \cdot \nabla u\, dx} + 2\alpha \int_\Omega \nabla u\cdot\nabla g\, dx + \alpha^2 \int_\Omega \nabla g\cdot\nabla g\, dx$$
$$- \cancel{\int_\Omega \nabla u\cdot\nabla u\, dx}$$

$$= \underbrace{2\alpha \int_\Omega \nabla u\cdot \nabla g\, dx}_{\text{Linear part}} + \underbrace{\alpha^2 \int_\Omega \nabla g \cdot \nabla g\, dx}_{\text{other part} \,=\, C|h|^2}$$

notice $\alpha = |h|$

$$\Rightarrow \quad \frac{\text{other part}}{|h|} \to 0 \quad |h| \to 0$$

because $\dfrac{C|h|^2}{|h|} = C|h| \underset{|h|\to 0}{\to} 0$!

dealing with

$$2\alpha \int_\Omega \nabla u \cdot \nabla g \, d\mu$$

$\nabla \cdot (g \nabla u)$ ← apply to

$\vec{V} = (V_1, V_2, V_3)$

$$\nabla \cdot \vec{V} = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial X_3}$$

$$\int_\Omega \nabla \cdot \vec{V} \, d\mu = \int_{\partial\Omega} \vec{V} \cdot \vec{n} \, \underbrace{d\sigma}_{\text{measure on } \partial\Omega}$$

$$\int_\Omega \nabla \cdot g \nabla u \, d\mu = \int_{\partial\Omega} \cancel{\underset{=0 \text{ on } \partial\Omega}{g} \nabla u \cdot \vec{n} \, d\sigma} \to 0$$

$$\int_\Omega \nabla g \cdot \nabla u \, d\mu + \int_\Omega g \underbrace{(\nabla \cdot \nabla u)}^{\Delta u} d\mu = 0$$

$$-\int_\Omega \Delta u\, g \, d\mu = \int_\Omega \nabla u \cdot \nabla g \, d\mu$$

example in $\mathbb{R}^3$

$$\nabla \cdot \nabla u = \nabla \cdot \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right)$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= u_{xx} + u_{yy} + u_{zz}$$

Δu

$$F(u+h) - F(u) =$$

$$-2\alpha \int \Delta u\, g \, d\mu \quad + o(h)$$

$$= \underbrace{-2 \int \Delta u \, h \, d\mu}_{\text{notice this is lin in } h} + o(h)$$

$$L(h) = -2\int \Delta u \, h \, d\mu$$

$$L(\alpha v + \beta w) = \alpha L(v) + \beta L(w)$$

$$-2\int \Delta u \, (\alpha v + \beta w) = \alpha\left(-2\int \Delta u \, v\right) + \beta\left(-2\int \Delta u \, w\right)$$

yes!! L is linear!!

$$L(h) = -2\int \underline{\Delta u} \, h \, d\mu$$

Joy of minimizing $F(u)$ given

$$u\big|_{\partial\Omega} = f$$

step 1: compute derivative of

$F \Rightarrow -2\int \underline{\Delta u} \, h \, d\mu$

step 2: notice if I pick

$$V = \frac{\Delta u}{|\Delta u|^2}$$

$u \to u + V$

$$-2 \int \Delta u \frac{-\Delta u}{|\Delta u|^2} \, d\mu$$

$$2 \int_\Omega \frac{(\Delta u)^2}{|\Delta u|^2} \, d\mu = 2 \operatorname{Vol}(\Omega)$$

$$= 2 \underbrace{\int_\Omega 1 \, d\mu}_{\operatorname{Vol}^n(\Omega)}$$