$V$ vector space

$|\cdot|$ norm on vector

example $L^2$ norm on $\mathbb{R}^n$

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$
$$= \sqrt{x \cdot x}$$

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A vector space with a norm that is complete as a metric space, is called a Banach Space

(look up the Scottish Cafe in Warsaw)

Ulam

$$|\alpha x| = |\alpha|\,|x|$$
$$|x+y| \leq |x| + |y|$$

M is an inner product if

write this as $\langle x, y\rangle_M$

$$x^T M y$$

$$(x_1\ x_2\ x_3)\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

if $\langle x, y\rangle_M$ is linear in each argument

$$\langle x, y\rangle_M = \langle y, x\rangle_M$$

and $\langle x, x\rangle_M \geq 0$ $= 0$ iff $x = 0$

$\Longleftrightarrow$ M is positive definite symmetric

$$0 < \lambda_{min} |x| \leq |Mx| \leq \lambda_{max} |x|$$

$$M = M^T \quad \text{(Because we are "real")}$$

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if it turns out that $|\cdot|_V$ has the form

$$|\cdot|_V = \sqrt{\langle \cdot, \cdot \rangle_M}$$

i.e. $|x|_V = \sqrt{\langle x, x \rangle_M}$ for some p.d.s matrix $M$

then $V$ is called a

$\rightarrow$ Hilbert Space

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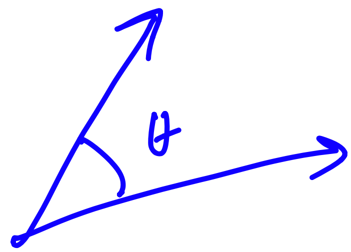


$$\theta_{v_1,v_2} \equiv \cos^{-1}\left(\frac{\langle v_1, v_2\rangle_M}{|v_1|\,|v_2|}\right)$$

$$\cos(\theta) = \frac{v_1 \cdot v_2}{|v_1|\,|v_2|}$$

$$\cos(\theta)\,|v_1|\,|v_2| = v_1 \cdot v_2$$

$$\cos(\theta) = \frac{v_1}{|v_1|} \cdot \frac{v_2}{|v_2|}$$

$$\cos^{-1}(\cos(\theta)) = \cos^{-1}\left(\frac{v_1 \cdot v_2}{|v_1|\,|v_2|}\right)$$

So: in any Hilbert space, we define the angle between two vectors to be

$$\boxed{\theta_{(V_1, V_2)} \equiv \cos^{-1}\left(\frac{\langle V_1, V_2\rangle_M}{|V_1|\,|V_2|}\right)}$$

$$V_1 \cdot V_2 = 0$$

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M is P.D.S.

$\Rightarrow \exists$ an orthogonal basis of eigenvectors

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$$F(x+h) - F(x) = Ah + \underbrace{g(h)}_{o(h)}$$

$$\underline{F(x+h) - F(x) - Ah} = \cancel{g(h)}$$

$$\frac{|F(x+h) - F(x) - Ah|}{|h|} \to 0$$

$$h \to 0$$

$$F : \mathbb{R} \to \mathbb{R}$$

$$\frac{F(x+h) - F(x)}{h} \to F'(x)$$

$$\frac{|F(x+h) - F(x) - \widetilde{F'(x)}\, h|}{|h|} \to 0$$

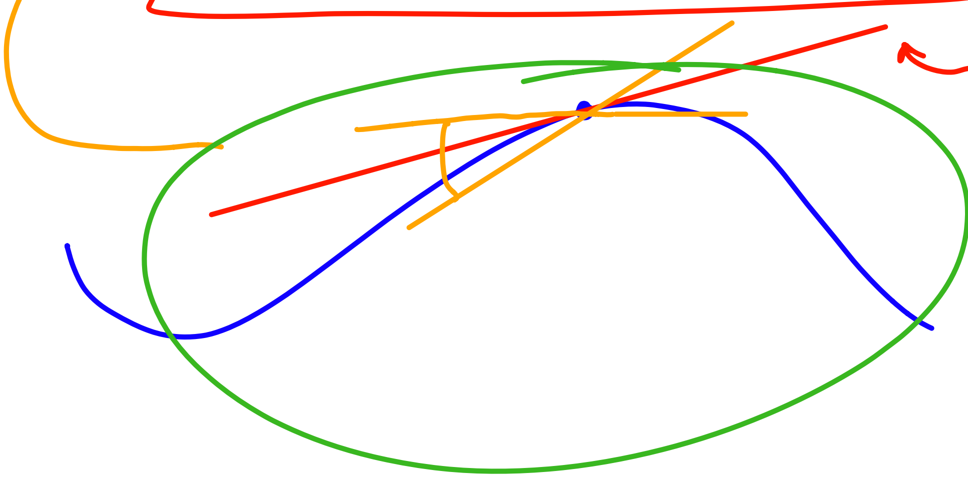
$$\frac{F(x+h) - F(x)}{h} \to \frac{F'(x)\,h}{h}$$

$$\frac{|F(x+h) - F(x) - F'(x)\,h|}{|h|} \to 0$$

$\underline{\Delta_x} F(h) \equiv F(x+h) - F(x)$

$$\boxed{\Delta_x F(h) = Ah + o(h)}$$

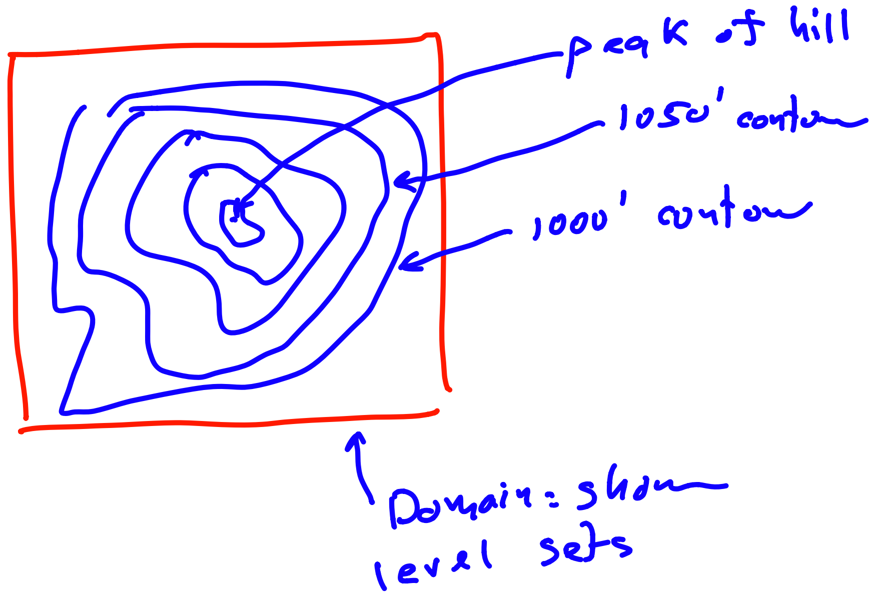
$$\boxed{\underline{\Delta_x F(h)} - \underline{Ah} \sim o(h)}$$

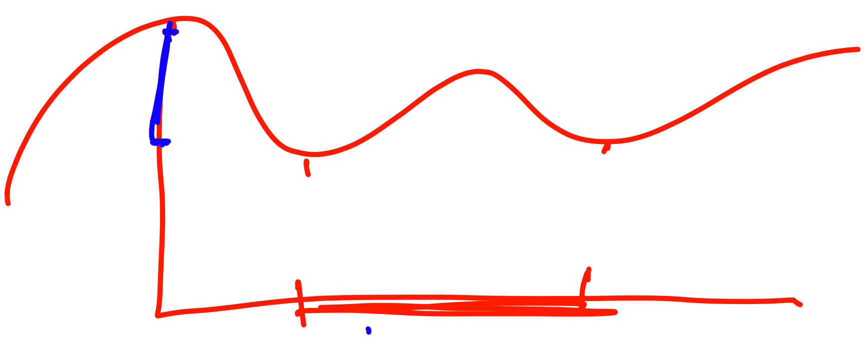


graph of derivation

$F: \mathbb{R}^n \to \mathbb{R}^m$

graph space $= \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$



$$F(u) = \int_\Omega \nabla u \cdot \nabla u \, dx$$

1. you know what gradients are
2. you know what this integral means
3. so you understand what this mapping the space of $C^2(\Omega)$ to $\mathbb{R}$ means
4. understanding the linear approx to $F$ at any given $u$ in the domain take a "gift" more thought.

Mission Find $\underbrace{A}_{\text{linear}} : C^2(\Omega) \to \mathbb{R}$

such that

will also be in $C^2(\Omega)$

$$F(u+h) - F(u) - A(h) \sim o(h)$$

Answer

$$\boxed{A(h) = \int_\Omega \Delta u \, h \, dx}$$

check

$$A(\alpha h + \beta w) =$$

$$\alpha A(h) + \beta A(w)$$

$F(u)$ $\quad F$: function space $\to \mathbb{R}$

$\frac{\delta F}{\delta u} \Rightarrow$ Euler-Lagrange Equation

$$\Delta u = 0$$

$$\underline{A}\ \underline{h}$$

$$\int \underline{\Delta u}\, h$$

$$\Delta u = 0$$