Question 1

Vector Space V $\longleftrightarrow$ Dual space V^*

continuous linear maps from $V \to \mathbb{R}$ or $V \to \mathbb{C}$

any element of V^* is called a co-vector

In the setting of $\mathbb{R}^n$ we usually call column vectors vectors & row-vectors, covectors

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In $\mathbb{R}^n$ $\quad V$ "=" V^*

$\mathbb{R}^n \approx (\mathbb{R}^n)^* \qquad v \in \mathbb{R}^n \leftrightarrow \phi \in (\mathbb{R}^n)^*$

$$\underline{v} \xrightarrow{J} \underbrace{\langle v, \cdot \rangle}_{\phi_v} \in (\mathbb{R}^n)^*$$

$$\underbrace{|\phi_v| \equiv \sup_{|w|=1} \frac{|\phi_v(w)|}{|w|}}_{\text{operator norm}}$$

$J : v \to \phi_v$

J is one to one
onto
Linear
isometric

$\left(\mathbb{R}^2 , \ |x|_1 \right) = |x_1| + |x_2|$

$\downarrow$ dual space

$\left(\mathbb{R}^2 , \ |x|_\infty \right) = \sup\{|x_1|, |x_2|\}$

Question 2

Banach vs Hilbert Spaces

(a) vector space $(V, \mathbb{F})$ $\leftarrow$ $(\mathbb{R}, \mathbb{C})$

(b) $| \ |_V$ norm on V

$$|\alpha w|_V = |\alpha| \, |w|_V$$

$$|w + u|_V \leq |w|_V + |u|_V$$

$$|w|_V = 0 \iff w = 0$$

(c) is it complete?

Cauchy sequences have limits.

$$P^* = \bigcup_{n=1}^{\infty} P_n$$

P_n = nth degree polynomials on $[0, 1]$

P^* with sup norm ... $|p| = \sup_{x \in [0,1]} |p(x)|$ is not complete!!

Because $\exists$ sequences $\{p_i\}_{i=1}^{\infty}$ in P^* $\ni$ p_i is cauchy but $p_i \to$ non-polynomial

Banach space: complete, normed vector space.

d) inner product on V

$\langle u, v \rangle_V$: bilinear

$\langle u, u \rangle \geq 0$ $\quad = 0 \Leftrightarrow u = 0$

and $|w|_V \equiv \sqrt{\langle w, w \rangle}$

Hilbert space: Banach space whose norm comes from an inner product

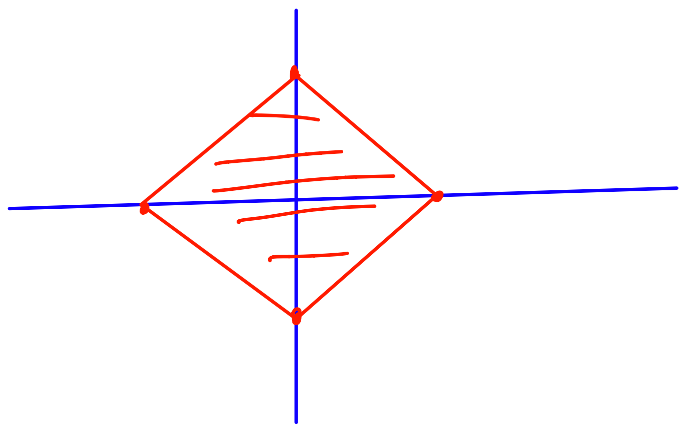
... have the look and feel of $\mathbb{R}^n$

Examples of

$B \setminus H$

① $\mathbb{R}^2, \ |\cdot|_1$

$$|w|_1 \equiv |w_1| + |w_2|$$



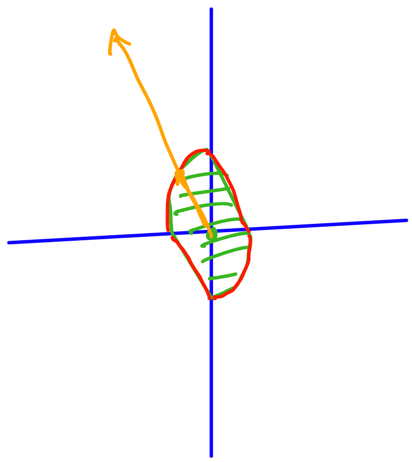
② $\mathbb{R}^n, \quad |\cdot|_p \quad p \neq 2$

$$|\cdot|_p : w \in \mathbb{R}^n = \left(\sum_{i=1}^{n} |w_i|^p \right)^{1/p}$$

$p = 1, 2, \infty$

③ (Inner products $\Longleftrightarrow$ symmetric positive definite matrices

(norms $\iff$ convex, symmetric w.r.t. origin, non-empty interior sets



Question 3

SVD

diagonal

$$M = U \Sigma V^T$$

orthogonal

orthogonal

$m > n$

$$n\underset{}{\overset{m}{[\,M\,]}} = n\overset{n}{[\,U\,]}\underbrace{\begin{bmatrix} \sigma_1 & & & \vdots & \\ & \sigma_2 & & \vdots & 0 \\ & & \ddots\ \sigma_n & \vdots & \end{bmatrix}}_{}\ m\overset{m}{\begin{bmatrix} & & \\ & V^T & \\ & & \end{bmatrix}}$$

non-negative

$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_n$

$Mu = \lambda u$

$\rightarrow Mu_i = \lambda_i u_i \qquad i = 1 \dots n$

$MU = U\Lambda \qquad U = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}$

$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

U orthogonal

$MU = U\Lambda$

$MUU^T = U\Lambda U^T$

$M = U\Lambda U^T$

$$\lambda_i = \alpha + i\beta$$
$$\lambda_j = \alpha - i\beta$$

$$\begin{array}{cccc} & [0 \ldots \sigma_i \ldots 0] & \begin{array}{c} i \\ \downarrow \\ [\;\ 0 \;\ 1 \;\ 0 \ldots 0] \end{array} & \\ M = & \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} & \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} & \begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix} \\ & & [0 \ldots 0 \;\ \sigma_i \;\ 0 \ldots 0] & \end{array}$$

$$\boxed{M v_i = \sigma_i u_i}$$

ith $\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$

$$M = \sum_{i=1}^{n} \sigma_i \underline{u_i v_i^T}$$

$$= \sum_{i=1}^{n} |u_i\rangle \sigma_i \langle v_i|$$